The Damped Pendulum

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Contents

1 The Damped Pendulum

The context under which the simple pendulum is typically discussed is not subject to energy dissipation, and hence, the energy remains a constant throughout its motion. In this essay we consider the motion of the pendulum in the presence of friction. Due to the presence of friction the energy of the pendulum no longer remains a constant, and hence, there is energy dissipation throughout the motion. Regardless of the initial conditions, the pendulum finally comes to rest at its downward vertical position. We can envision friction arising in two main forms: it can be present in the pivot, and it can be present due to air drag (if the pendulum moves in air, that is, of course; in general, it can be any fluid medium). For the present discussion let us only consider the air drag which can be taken as being proportional to the instantaneous speed for relatively slower swings. We can then write the frictional damping force (F_f) as $F_f = av$ where a (> 0) is a constant and v is the instantaneous speed. Denoting the angle that the pendulum makes with the downward vertical as θ (positive when measured counterclockwise), $v = l\dot{\theta}$ where l is the length of the pendulum measured from the pivot point; therefore, $F_f = al\dot{\theta}$. Here a single (double) overdot refers to the first (second) derivative with respect to time. Applying Newton's second law, we obtain,

$$-mg\sin\theta - al\dot{\theta} = ml\ddot{\theta},$$

where m is the mass of the pendulum concentrated in the bob and g is the constant gravitational acceleration. The equation of motion can then be written as,

$$\ddot{\theta} = -\left(\frac{g}{l}\right)\sin\theta - \left(\frac{a}{m}\right)\dot{\theta}.$$

We can recognize the factor g/l as the square of the angular frequency ω_0^2 of the undamped pendulum. Let us use 2α to denote the constant a/m and take $\alpha > 0$ to be the damping constant (the reason for the factor 2 is for convenience that would simplify our expressions later). The equation of motion for the damped pendulum can thus be written in its final form as,

$$\boxed{\ddot{\theta} = -\omega_0^2 \sin\theta - 2\alpha\dot{\theta}.} \tag{1}$$

2 The Linear Damped Pendulum

The presence of the $\sin\theta$ term in (1) makes the equation of motion nonlinear, and hence, impossible to find a solution in closed form. However, just as with the undamped pendulum for small swings, many interesting behaviors of the pendulum can be surfaced by considering small librations for which $\sin\theta \approx \theta$, in which case closed form solutions can be found. With the linear approximation, the equation of motion (1) can be written as,

$$\boxed{\ddot{\theta} + 2\alpha\dot{\theta} + \omega_0^2\theta = 0.} \tag{2}$$

Our goal now is to express θ as a function of time t satisfying the above expression subject to initial conditions. Toward this, let us posit a solution of the form $\theta(t) = e^{rt}$, where r stands for the 'roots' to be found. Given that the equation to be solved is a second order linear differential equation, we expect there to be two linearly independent solutions (hence two roots of r), which can be superimposed to form the full solution. If $\theta(t) = e^{rt}$, then it follows that $\dot{\theta} = r \cdot e^{rt}$ and $\ddot{\theta} = r^2 \cdot e^{rt}$. Substituting these expressions in (2) we obtain,

$$\left(r^2 + 2\alpha r + \omega_0^2\right) e^{rt} = 0.$$

For this equation to be satisfied for all t, it must hold true that,

$$r^2 + 2\alpha r + \omega_0^2 = 0. \tag{3}$$

This quadratic equation in r is the characteristic equation of (2) and carries two roots r_1 and r_2 which are easily found to be,

$$r_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \tag{4}$$

$$r_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}. \tag{5}$$

Given the simpler-looking forms for r_1 and r_2, the reader may now appreciate the presence of the factor 2 in the equation of motion. The general solution can now be expressed as,

$$\theta(t) = Be^{r_1 t} + Ce^{r_2 t}, \tag{6}$$

where B and C are constants (which are generally complex numbers). The specific behavior of the linear damped pendulum then depends on whether $\alpha^2 - \omega_0^2$ is negative ($\alpha^2 < \omega_0^2$: underdamped), zero ($\alpha^2 = \omega_0^2$: critically damped), or positive ($\alpha^2 > \omega_0^2$: overdamped). We study each of these scenarios in turn.

2.1 Underdamped Motion: $\alpha^2 < \omega_0^2$

To study the case of underdamping, where $\alpha^2 < \omega_0^2$, let us first introduce a new frequency $\omega^2 = \omega_0^2 - \alpha^2$. Thus, this new frequency ω is less than that of ω_0. Since in this case the discriminant $\alpha^2 - \omega_0^2 < 0$, the square root of the discriminant becomes a complex number. However, since $\omega^2 > 0$, ω is a real number. With $i^2 = -1$, the two roots can now be written as,

$$r_1 = -\alpha + i\omega,$$
$$r_2 = -\alpha - i\omega.$$

The real part $(-\alpha)$ of the roots signifies the decay of the pendulum's motion while the imaginary parts $(\pm i\omega)$ of the roots signify its oscillations. So the resultant motion is a combination of the two giving rise to oscillations that decay over time. Substituting the latest expressions for the roots in (6), we obtain

$$\theta(t) = e^{-\alpha t}\left(Be^{i\omega t} + Ce^{-i\omega t}\right).$$

We immediately see that since $\alpha > 0$, as $t \to \infty$, $\theta \to 0$. Since $e^{\pm i\omega t} = \cos\omega t \pm i \sin\omega t$, substituting and collecting the real and imaginary parts result in,

$$\theta(t) = e^{-\alpha t}\left[(B+C)\cos\omega t + i(B-C)\sin\omega t\right].$$

Now, let us consider a right triangle with hypotenuse A, opposite side length $(B+C)$, and the adjacent side length $i(B-C)$ with the angle between the adjacent and the hypotenuse Φ. Thus, $\sin\Phi = (B+C)/A$ and $\cos\Phi = i(B-C)/A$. The above expression can then be written as,

$$\begin{aligned}\theta(t) &= Ae^{-\alpha t}\left[\frac{(B+C)}{A}\cos\omega t + \frac{i(B-C)}{A}\sin\omega t\right] \\ &= Ae^{-\alpha t}\left(\sin\omega t\cdot\cos\Phi + \cos\omega t\cdot\sin\Phi\right) \\ \theta(t) &= Ae^{-\alpha t}\cdot\sin(\omega t + \Phi)\end{aligned}$$

In this expression, A denotes the amplitude, α the damping constant, ω the effective frequency, and Φ the phase. Differentiating the above expression with respect to time yields the instantaneous angular velocity $\dot{\theta}(t)$. We collect the expressions for $\theta(t)$ and $\dot{\theta}(t)$ below:

$$\theta(t) = Ae^{-\alpha t} \cdot \sin(\omega t + \Phi) \tag{7}$$
$$\dot{\theta}(t) = Ae^{-\alpha t} \left[\omega \cdot \cos(\omega t + \Phi) - \alpha \cdot \sin(\omega t + \Phi)\right] \tag{8}$$

The amplitude A and the phase Φ are determined by the initial conditions at $t = 0$. Suppose $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$ are the initial conditions. Then,

$$\theta(0) = \theta_0 = A \cdot \sin \Phi \tag{9}$$
$$\dot{\theta}(0) = \dot{\theta}_0 = A \left[\omega \cdot \cos \Phi - \alpha \cdot \sin \Phi\right] \tag{10}$$

dividing (10) by (9) and solving for Φ, we determine the phase to be,

$$\Phi = \cot^{-1}\left[\frac{1}{\omega}\left(\frac{\dot{\theta}_0}{\theta_0} + \alpha\right)\right].$$

The value for Φ can then be substituted back into (9) to determine the amplitude A, yielding,

$$A = \frac{\theta_0}{\sin\left[\cot^{-1}\left[\frac{1}{\omega}\left(\frac{\dot{\theta}_0}{\theta_0} + \alpha\right)\right]\right]}.$$

Figures 1 and 2 show the decay in the oscillations in both the angle and the angular velocity against time for a specific set of parameters. Figure 3 shows the effect of the damping constant in the decay of oscillations; a larger damping constant results in a swifter decay of oscillations.

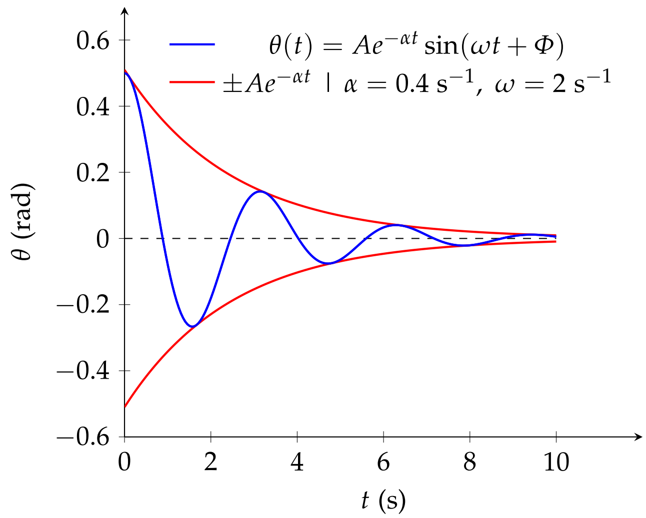

Figure 1: The underdamped decay of the oscillations in the angle θ of the damped linear pendulum against time t (blue curve). The oscillations are enveloped by the decay functions (red curves). For the case shown, $\alpha = 0.4\ \mathrm{s}^{-1}$, $\omega = 2\ \mathrm{s}^{-1}$. The initial conditions are $\theta_0 = 0.5$ rad and $\dot{\theta}_0 = 0\ \mathrm{rad\ s}^{-1}$.

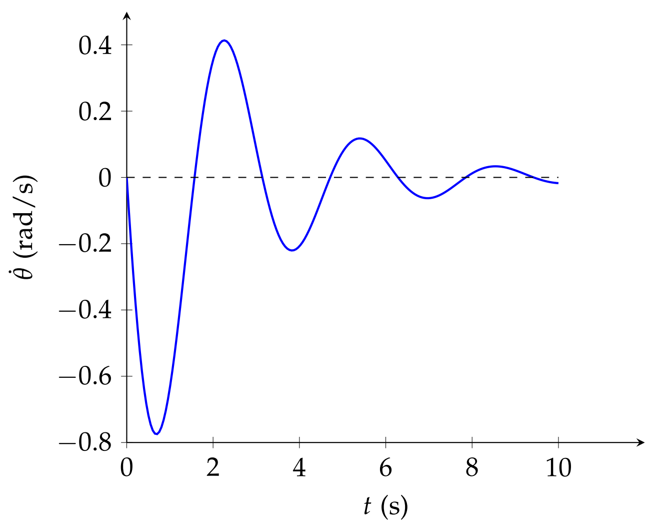

Figure 2: The underdamped decay in angular velocity $\dot{\theta}$ of the damped linear pendulum against time t. For the case shown, $\alpha = 0.4\ \mathrm{s}^{-1}$, $\omega = 2\ \mathrm{s}^{-1}$. The initial conditions are $\theta_0 = 0.5$ rad and $\dot{\theta}_0 = 0\ \mathrm{rad\ s}^{-1}$.

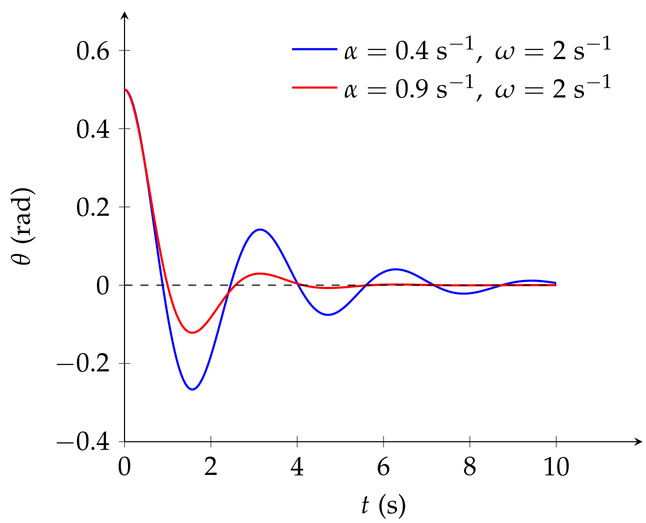

Figure 3: The underdamped decay of the oscillations in the angle θ of the damped linear pendulum against time t for two different damping constants $\alpha = 0.4\ \mathrm{s}^{-1}$ (blue) and $\alpha = 0.9\ \mathrm{s}^{-1}$ (red) with $\omega = 2\ \mathrm{s}^{-1}$. Larger damping constants lead to swifter decays in oscillations. The initial conditions are the same for both cases: $\theta_0 = 0.5$ rad and $\dot{\theta}_0 = 0$ rad s^{-1}.

Figure 4 shows the phase portrait of a linear underdamped pendulum for two different damping constants but having the same ω (hence, not the same ω_0) and starting with the same initial conditions. The phase curves spiral from the starting point and terminate at the origin $(0, 0)$ which correspond to the pendulum hanging vertically down at rest. This is the ultimate state of the pendulum given the energy dissipation due to damping. Therefore the origin of phase space is an attracting point since all phase curves of the linear underdamped pendulum terminate there. Figure 5 shows a multitude of phase curves having the same damping constant α and same ω_0 (and hence, the same ω) but starting at different initial conditions. Note that, in Figure 5, since the parameters α and ω_0 are fixed for all initial conditions, the phase curves cannot intersect since that would violate the uniqueness of the solutions to the associated equation of motion. In other words, if the phase curves cross, then the pendulum has to "decide" which path to take at the cross roads, which it cannot do.

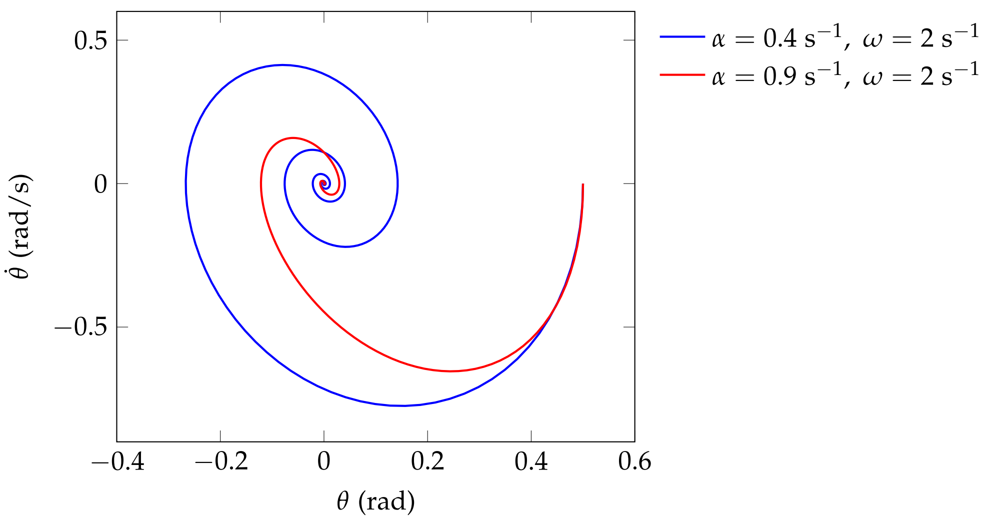

Figure 4: The phase curves of the linear underdamped pendulum for two sets of parameters. The initial conditions, $\theta_0 = 0.5$ rad, $\dot{\theta}_0 = 0$ rad s^{-1}, are the same for both. Both phase curves terminate at the attracting point $(0, 0)$.

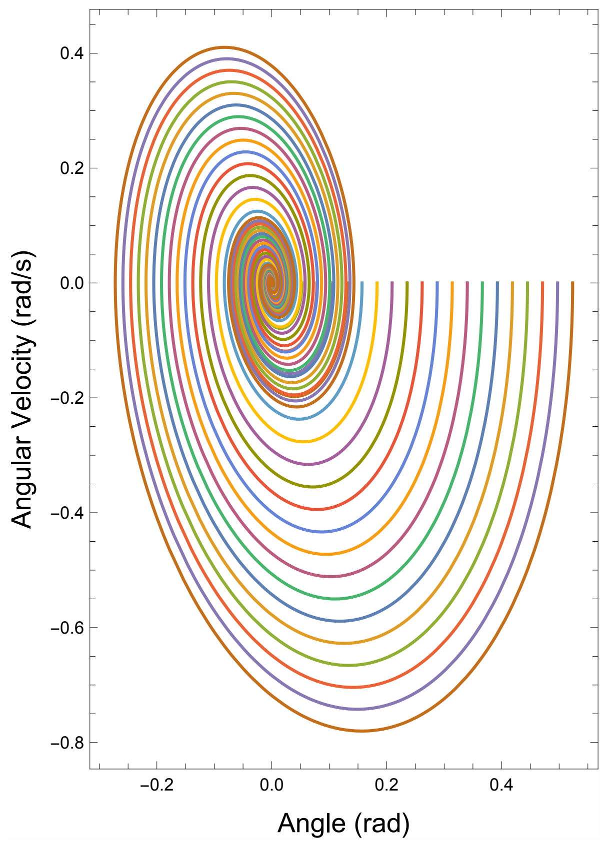

Figure 5: Phase curves for the linear underdamped pendulum starting at different initial conditions. For all curvea $\alpha = 0.4\,\mathrm{s}^{-1}$ and $\omega_0 = 2\,\mathrm{s}^{-1}$. All phase curves terminate at the origin $(0, 0)$, which is the attracting point. Since the parameters α and ω_0 are fixed for all initial conditions, the phase curves cannot intersect since that would violate the uniqueness of the solutions to the associated equation of motion. In other words, if the phase curves cross, then the pendulum has to "decide" which path to take at the cross roads, which it cannot do.

We can add the time dimension to a phase curve of a linear underdamped pendulum and visualize its evolution in time as shown in Figure 6.

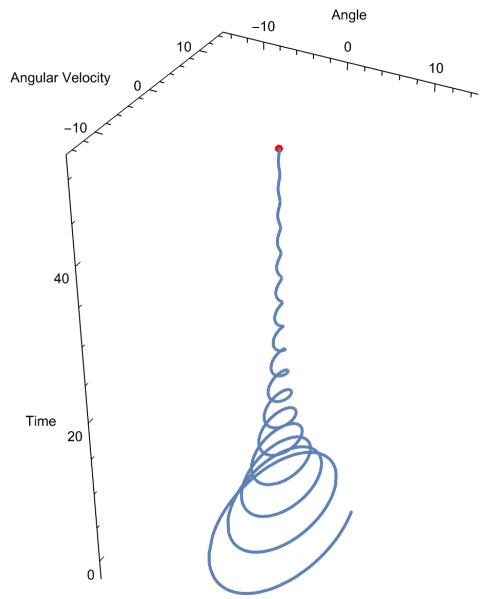

Figure 6: The time (in seconds) evolution of a phase curve (blue) of a linear underdamped pendulum with $\alpha = 0.1$ s$^-1$, $\omega_0 = 2$ s$^-1$, and initial conditions $\theta_0 = 2\pi$ rad, $\dot{\theta}_0 = 2\pi$ rad s^{-1}. The red dot shows the attracting point at $\theta = \dot{\theta} = 0$ on the angle-angular velocity phase plane where the motion terminates due to energy loss.

Before moving further we wish to address a potential befuddlement that the reader may harbor regarding the expressions for $\sin \Phi$ and $\cos \Phi$, which were derived from a right triangle having side lengths $(B + C)$, $i(B - C)$, and A. We have also commented earlier that B and C are complex numbers in general. Therefore, it may appear as if the lengths $(B + C)$ and $i(B - C)$ are not real numbers, which they must be for the sine and cosine values to make sense. Here we wish to demonstrate that all is well with regards to these lengths. To see this, let us claim that $A_1 = (B + C)$ and $A_2 = i(B - C)$ are real numbers. Solving for B and C, we obtain $B = \frac{1}{2}(A_1 - iA_2)$ and $C = \frac{1}{2}(A_1 + iA_2)$. Thus B and C are complex conjugates of each other, so one can be obtained from the other by letting $i \to -i$. (We will denote the complex conjugate of a complex number z by z^*; therefore, $B = C^*$ and $B^* = C$.) This indeed must be the case for the solution $\theta(t) = e^{-\alpha t}(Be^{i\omega t} + Ce^{-i\omega t})$ to hold. That is, if $\theta(t) = e^{-\alpha t} \cdot Be^{i\omega t}$ is a solution, then its complex conjugate $\theta(t) = e^{-\alpha t} \cdot B^* e^{-i\omega t}$ must also be a solution. But as shown above, $B^* = C$, and therefore $\theta(t) = e^{-\alpha t} \cdot Ce^{-i\omega t}$ is a solution. The superposition of both these solutions is then also a solution for $\theta(t)$ with $\theta(t)$ real. Thus the lengths $(B + C)$ and $i(B - C)$ are real numbers and are consistent with the solution to the linear underdamped pendulum.

2.2 Critically Damped Motion: $\alpha^2 = \omega_0^2$

In the case of the critically damped motion the solutions take the form,

$$\theta(t) = e^{-\alpha t}\left(B + Ct\right) \tag{11}$$

$$\dot{\theta}(t) = e^{-\alpha t}\left[C - \alpha(B + Ct)\right] \tag{12}$$

The constants B and C can be found using the initial conditions θ_0 and $\dot{\theta}_0$ at time $t = 0$ resulting in,

$$B = \theta_0, \quad C = \dot{\theta}_0 + \alpha\theta_0.$$

Let us now discuss the origin of the solution (11) closely. In the critically damped case the two roots given in (4) and (5) are the same with $r_1 = r_2 = -\alpha$. Certainly, one of the solutions to the equation of motion (2) can still be claimed to be of the form $Be^{-\alpha t}$. We now need to find a second solution that is linearly independent to the first; that is, the second solution must not be a constant multiple of the first. With the two roots being the same, therefore, the second solution cannot take the form $Ce^{-\alpha t}$. To find this second linearly independent solution, let us put the equation of motion in the form

$$\ddot{\theta} + b\dot{\theta} + c\theta = 0, \tag{13}$$

where $b = 2\alpha$ and $c = \omega_0^2$. The characteristic equation then takes the form

$$r^2 + br + c = 0. \tag{14}$$

Given the fact that if the two roots of a quadratic equation are r_1 and r_2, then their sum is the negative of the ratio of the coefficient of the linear term (b in this case) to the coefficient of the quadratic term (1 in this case), and their product is the positive of the ratio of the coefficient of the constant term (c in this case) to the coefficient of the quadratic term, we have,

$$r_1 + r_2 = -b, \quad r_1 r_2 = c.$$

Since in the critically damped case $r_1 = r_2$, we have $2r_1 = -b$ and $r_1^2 = c$. The approach to finding a second linearly independent solution to that of $Be^{r_1 t}$ is to perturb the root r_1. That is, we take the first root to be $r_1' = r_1$ and we let the second root to be $r_2' = r_1 + \rho$, where $\rho > 0$ is an infinitesimally small value. Thus r_1' and r_2' differ infinitesimally. Therefore, their sum and the product becomes,

$$r_1' + r_2' = 2r_1 + \rho = -b + \rho, \quad r_1' r_2' = r_1(r_1 + \rho) = r_1^2 + r_1\rho = c + r_1\rho.$$

Now, we understand that the roots r_1' and r_2' can no longer be the roots of the original equation of motion (13), but to a slightly perturbed equation of motion. Using the relationship between the roots and the coefficients of a quadratic equation stated above, the characteristic equation of the perturbed equation of motion becomes $r'^2 - (r'_1 + r'_2)r' + r'_1 r'_2 = 0$, or equivalently,

$$r'^2 + (b - \rho)r' + (c + r_1\rho) = 0. \tag{15}$$

Hence, the perturbed equation of motion that would satisfy this characteristic equation takes the form,

$$\ddot{\theta}_\rho + (b - \rho)\dot{\theta}_\rho + (c + r_1\rho)\theta_\rho = 0, \tag{16}$$

The subscript ρ signifies that the solution to $\theta(t)$ is now dependent on ρ. In the limit $\rho \to 0$ we recover (13) from (16).

Given the two new roots $r'_1 = r_1$ and $r'_2 = r_1 + \rho$, we can now posit a solution to $\theta_\rho(t)$ as,

$$\theta_\rho(t) = \frac{1}{\rho}\left[e^{(r_1+\rho)t} - e^{r_1 t}\right]. \tag{17}$$

To verify that this indeed is a solution of (16) let us find the first and second derivates of $\theta_\rho(t)$ with respect to time and substitute in (16). The derivatives are,

$$\dot{\theta}_\rho = \frac{1}{\rho}\left[(r_1+\rho)e^{(r_1+\rho)t} - r_1 e^{r_1 t}\right], \quad \ddot{\theta}_\rho = \frac{1}{\rho}\left[(r_1+\rho)^2 e^{(r_1+\rho)t} - r_1^2 e^{r_1 t}\right].$$

Substitution in (16) and rearrangement results in (for the left hand side of (16)),

$$-\frac{1}{\rho}\left(r_1^2 + br_1 + c\right)e^{r_1 t} + \frac{1}{\rho}\left[(r_1+\rho)^2 + (b-\rho)(r_1+\rho) + (c + r_1\rho)\right]e^{(r_1+\rho)t} + \frac{1}{\rho}\left(r_1\rho - r_1\rho\right)e^{r_1 t}.$$

The expression in the first bracket is zero since r_1 is a root of (14); similarly, the expression in the second bracket is zero since $r_1 + \rho = r'_1$ is a root of (15); and the last expression vanishes as well. Therefore the solution (17) satisfies the second order differential equation (16). However, what we really are interested in is the solution to (13), which can be obtained in the limit $\rho \to 0$. Thus,

$$\theta(t) = \lim_{\rho\to 0}\theta_\rho(t) = \lim_{\rho\to 0}\frac{1}{\rho}\left[e^{(r_1+\rho)t} - e^{r_1 t}\right].$$

Expanding the exponential terms in the infinite series form, we have,

$$e^{(r_1+\rho)t} = 1 + \frac{(r_1+\rho)t}{1!} + \frac{(r_1+\rho)^2 t^2}{2!} + \frac{(r_1+\rho)^3 t^3}{3!} + \cdots, \quad e^{r_1 t} = 1 + \frac{r_1 t}{1!} + \frac{r_1^2 t^2}{2!} + \frac{r_1^3 t^3}{3!} + \cdots.$$

Substituting these in the limit expression, we observe that the terms that do not carry ρ in the numerators cancel, and that the terms that are second or higher order in ρ goes to zero when $\rho \to 0$. This finally leaves us with an expression of the form,

$$\theta(t) = t\left(1 + \frac{r_1 t}{1!} + \frac{(r_1 t)^2}{2!} + \cdots\right) = te^{r_1 t}.$$

Therefore, the solution that is linearly independent to that of $e^{r_1 t}$ is $t \cdot e^{r_1 t}$. Since $r_1 = -\alpha$ in the case of the critically damped pendulum, this justifies the solution written in (11).

2.3 Overdamped Motion: $\alpha^2 > \omega_0^2$

In the case of over damped motion, since $\alpha^2 - \omega_0^2 > 0$, a positive real constant ω^2 can be directly defined as $\omega^2 = \alpha^2 - \omega_0^2$. Therefore, the two roots r_1 and r_2 become,

$$r_1 = -\alpha + \omega, \quad r_2 = -\alpha - \omega.$$

Thus the solutions take the form,

$$\theta(t) = e^{-\alpha t}\left(Be^{\omega t} + Ce^{-\omega t}\right) \tag{18}$$

$$\dot{\theta}(t) = e^{-\alpha t}\left[(\omega - \alpha) \cdot Be^{\omega t} - (\omega + \alpha) \cdot Ce^{-\omega t}\right] \tag{19}$$

Again, the constants B and C can be found using the initial conditions θ_0 and $\dot{\theta}_0$ at time $t = 0$ resulting in,

$$B = \frac{1}{2}\left[\theta_0 + \frac{1}{\omega}(\dot{\theta}_0 + \alpha\theta_0)\right], \quad C = \frac{1}{2}\left[\theta_0 - \frac{1}{\omega}(\dot{\theta}_0 + \alpha\theta_0)\right].$$

Figure 7 shows the change in the angle over time for underdamped, critically damped, and overdamped scenarios for the linear pendulum for a specific set of parameters. Figure 8 shows the corresponding motion in the phase plane.

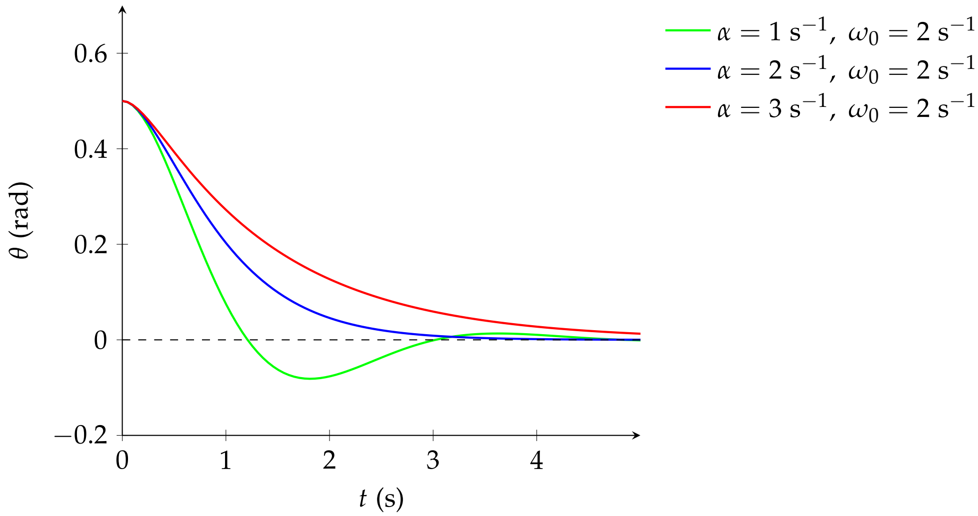

Figure 7: The underdamped (green), critical (blue), and overdamped (red) decays in the angle θ of the damped linear pendulum against time t. The initial conditions are the same for all cases: $\theta_0 = 0.5$ rad and $\dot{\theta}_0 = 0$ rad s^{-1}.

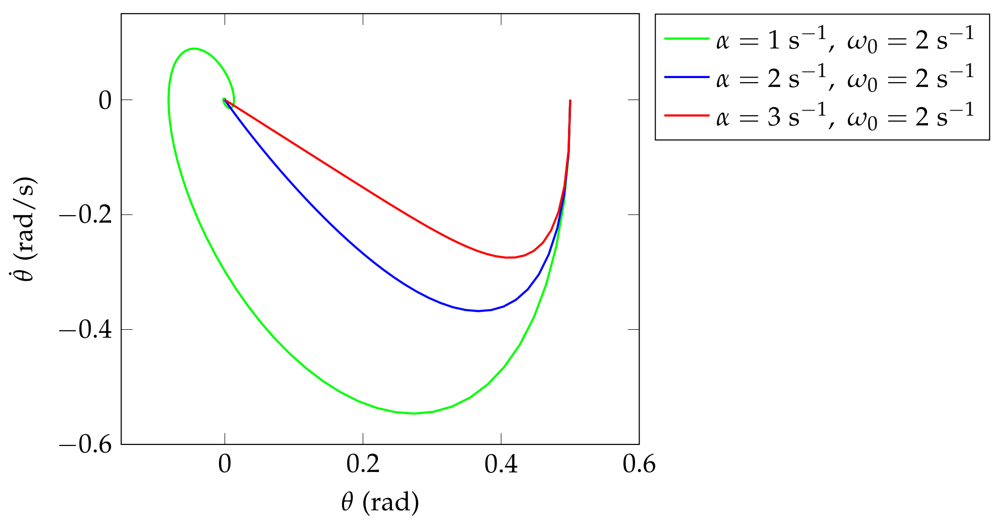

Figure 8: The phase curves for underdamped (green), critical (blue), and overdamped (red) cases of the damped linear pendulum. The initial conditions $\theta_0 = 0.5$ rad, $\dot{\theta}_0 = 0$ rad s^{-1}, are the same for all cases. Due to energy loss all phase curves terminate at the attracting point $(0, 0)$, which corresponds to the pendulum hanging at rest in the downward vertical position.

3 The NonLinear Damped Pendulum

In the previous section we analyzed the motion of the damped pendulum in the linear approximation where $\sin\theta$ was approximated by θ. We now remove this approximation and consider the motion of the damped pendulum in its full glory. We remind ourselves that the equation of motion of the damped pendulum is given by (1):

$$\ddot{\theta} = -\omega_0^2 \sin\theta - 2\alpha\dot{\theta}.$$

Just as in the case of the simple pendulum with no damping, there is no closed form solution to the above equation, and therefore, we must resort to numerical integration. Figure 9 shows the resulting phase portrait of the damped pendulum after numerical integration for a multitude of initial conditions but for the same set of parameters.

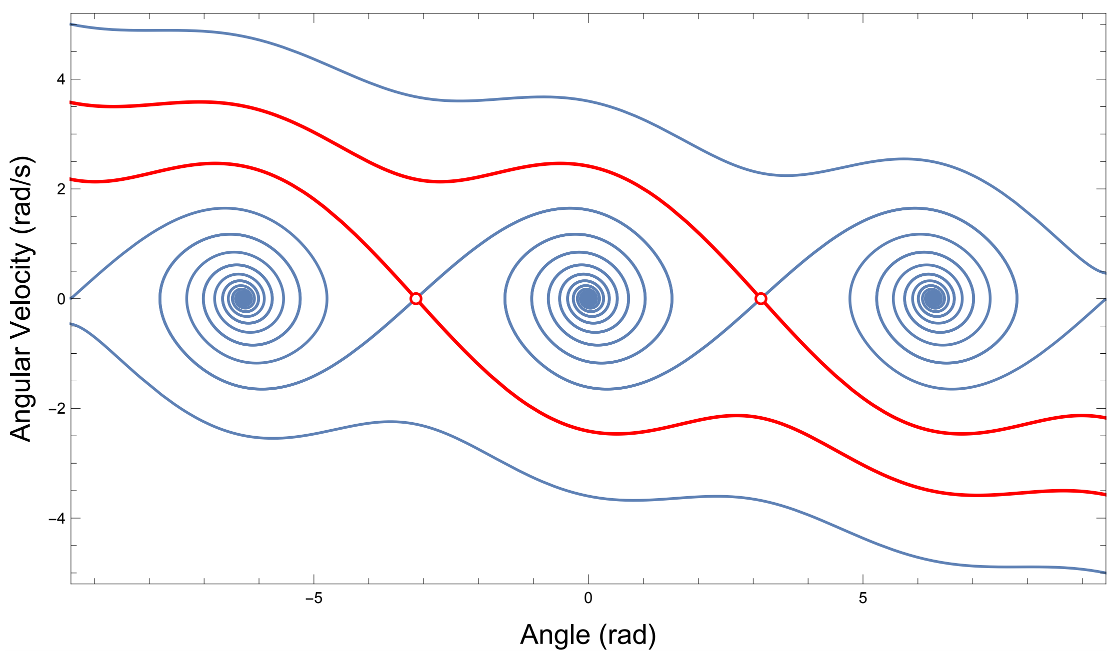

Figure 9: Phase curves for the damped pendulum (without the linear approximation) starting at different initial conditions. For all curves $\alpha = 0.1\ \mathrm{s}^{-1}$ and $\omega_0 = 1\ \mathrm{s}^{-1}$. The red phase curves are the separatrices for the stable attractor point at the origin $(0,0)$ and bound the basin of attraction for that point. The unstable points marked in open red circles are at $(\pm\pi, 0)$. Two other stable attractor points are shown at $\pm 2\pi$.

Figure 9 shows that attractor points occur at angles $\pm n_e\pi$ where n_e are even integers; similarly, unstable critical points occur at angles $\pm n_o\pi$ where n_o are odd integers. The two red phase curves show the separatrices for the attractor point at the origin $(0,0)$; these separate the damped librations from the full rotations with respect to the attractor point at the origin. Any initial condition that falls between the shown separatrices will result in the damped pendulum ultimately settling at, and hence, attracted to, the origin (which is the rest position of the pendulum hanging vertically down). The separatrices belonging to a given attractor point goes through the unstable critical points on either side of the attractor point (one seperatrix through each unstable critical point). The phase space region bounded by the separatrices related to a given a attractor point is called the *Basin of Attraction* for that point. In Figure 9 there are three basins of attractions corresponding to the three attractor points shown.

If the pendulum is brought to an unstable equilibrium by rotating it counterclockwise by 180° or π (rad), then the pendulum will be at the $(\pi, 0)$ position. This is the situation where the pendulum will be hanging vertically 'upside down'. From this point, if the pendulum is given a slight nudge to the left, then it will eventually settle at the attractor point at $(2\pi, 0)$. Instead, if it is given a slight nudge to the right, then it will eventually settle at the attractor point at the origin. The separatrix that arrives at $(\pi, 0)$ from the side of positive angular velocity settles there in unstable equilibrium; the same applies to the separatrix that arrives at $(\pi, 0)$ from the side of negative angular velocity. Both lead to the scenario that the pendulum hangs vertically 'upside down' at $(\pi, 0)$, an extremely delicate scenario.

To analyze the behavior of the damped pendulum more quantitatively, let us first establish the positions of the stable and unstable critical points, which from the phase portrait above, we know are at $n\pi$ where n is either an even or an odd integer. For this, let us breakdown the second order differential equation of motion into two first order equations as follows:

$$\frac{d\theta}{dt} = \nu \tag{20}$$

$$\frac{d\nu}{dt} = -2\alpha\nu - \omega_0^2 \sin\theta \tag{21}$$

Given that the pendulum is in equilibrium at critical points, the angular velocity and angular acceleration must both vanish. Hence $\frac{d\theta}{dt}$ and $\frac{d\nu}{dt}$ are 0. Therefore, it follows from (20) that $\nu = 0$, and from (21) that $\sin\theta = 0$. The latter can only be satisfied if θ are integer multiples of π. Thus, $\theta = n\pi$ where n is an integer. Hence, the critical points occur at integer multiples of π.

Let us now zoom in on near a critical point $c = n\pi$ and look at the behavior of the phase curves near that point. For this let us first expand $\sin\theta$ around c using the Taylor expansion. For a function $f(\theta)$, the Taylor expansion about a point c is given by,

$$f(\theta) = f(c) + \frac{(\theta - c)}{1!} f'(\theta)\Big|_{\theta=c} + \frac{(\theta - c)^2}{2!} f''(\theta)\Big|_{\theta=c} + \cdots,$$

where $f'(\theta)$ is the first order differential of $f(\theta)$ with respect to θ and $f''(\theta)$ is the second order differential of $f(\theta)$ with respect to θ, and so on. Since $f(\theta) = \sin\theta$, using the expansion up to the first order (that is, to linear order) about the critical point c, we get,

$$\sin\theta = \sin(c) + (\theta - c)\cos(c) = 0 + (\theta - c)\cos(c)$$

since $\sin(c) = 0$ for $c = n\pi$ for any integer n. Now, about critical points where $n = n_e$ is an even integer, $\cos(c) = \cos(n_e\pi) = 1$ and about critical points where $n = n_0$ is an odd integer, $\cos(c) = \cos(n_o\pi) = -1$. Therefore, the linear approximation to sine at the respective critical points become,

$$\sin\theta = (\theta - n_e\pi) \quad \text{for } n_e \text{ even integer}$$
$$\sin\theta = -(\theta - n_o\pi) \quad \text{for } n_o \text{ odd integer}$$

The first order form of the equations of motion then becomes

$$\frac{d\theta}{dt} = v \tag{22}$$
$$\frac{dv}{dt} = -2\alpha v - \omega_0^2(\theta - n_e\pi) \quad \text{for } n_e \text{ even integer} \tag{23}$$
$$\frac{dv}{dt} = -2\alpha v + \omega_0^2(\theta - n_o\pi) \quad \text{for } n_o \text{ odd integer} \tag{24}$$

Figure 10 shows the behavior of the phase flow near the stable critical point $(2\pi, 0)$ according to the full equation of motion [given by (20) and (21)] and the linear approximation [given by (22) and (23)]. Judging by the figures side by side, we see that the linear approximation near the stable critical point (2π, hence $n_e = 2$ in this case) works very well and agrees with the full solution. Similarly, Figure 11 shows the behavior of the phase flow near the unstable critical point $(\pi, 0)$ according to the full equation of motion [given by (20) and (21)] and the linear approximation [given by (22) and (24)]. We again see that the linear approximation works well in comparison to the full solution in describing the motion near the unstable critical point (π, hence $n_o = 1$ in this case).

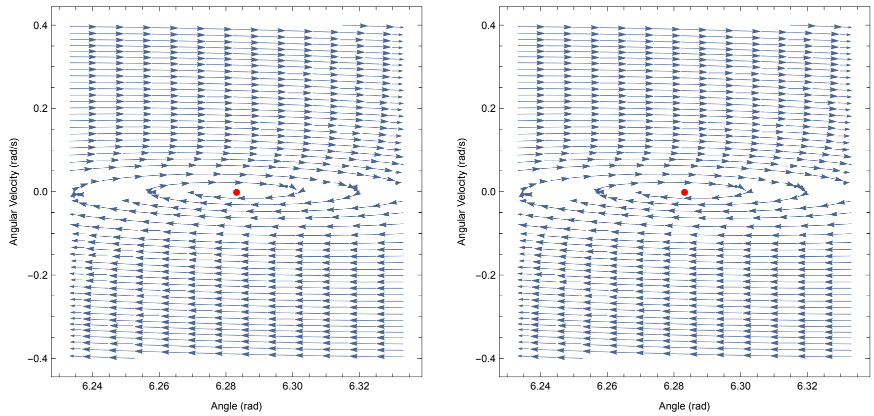

Figure 10: Left: The behavior of the phase flow near the stable critical point $(2\pi, 0)$ according to the full equation of motion [given by (20) and (21) with $\alpha = 0.1\ \mathrm{s}^{-1}$ and $\omega_0 = 1\ \mathrm{s}^{-1}$]. Right: The behavior of the phase flow near the same stable critical point per the linear approximation [given by (22) and (23)]. The phase flow near the stable equilibrium is well-described by the linear approximation.

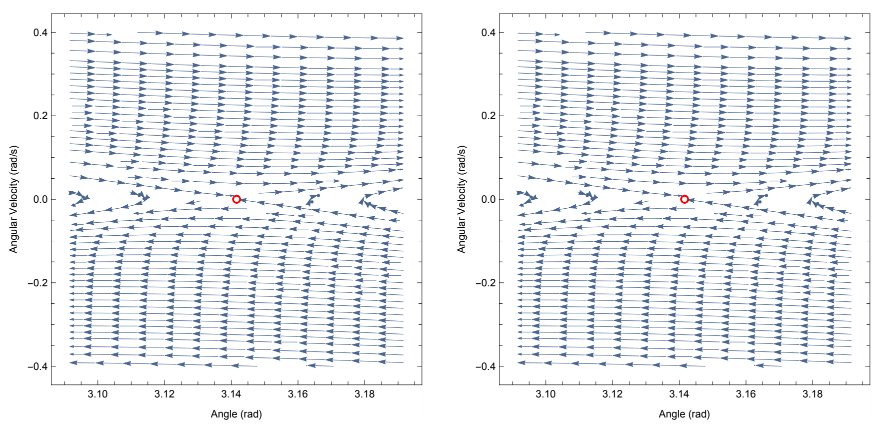

Figure 11: Left: The behavior of the phase flow near the unstable critical point $(\pi, 0)$ according to the full equation of motion [given by (20) and (21) with $\alpha = 0.1\ \mathrm{s}^{-1}$ and $\omega_0 = 1\ \mathrm{s}^{-1}$]. Right: The behavior of the phase flow near the same unstable critical point per the linear approximation [given by (22) and (24)]. The phase flow near the unstable equilibrium is well-described by the linear approximation.

Let us now turn to finding the explicit solutions to the linearly approximated equations of motion above. Near the stable equilibrium points the behavior of the phase curves are determined by (22) and (23). These are equivalent to the second order differential equation of motion $\ddot{\theta} + 2\alpha\dot{\theta} + \omega_0^2(\theta - n_e\pi) = 0$. Let $\lambda_e = \theta - n_e\pi$. Then $\dot{\theta} = \dot{\lambda}_e$ and $\ddot{\theta} = \ddot{\lambda}_e$. The equation of motion then becomes: $\ddot{\lambda}_e + 2\alpha\dot{\lambda}_e + \omega_0^2\lambda_e = 0$. As before, let us propose a solution of the form $\lambda_e = e^{rt}$. Then the characteristic equation becomes $r^2 + 2\alpha r + \omega_0^2 = 0$. But this is the same equation as (3). Therefore, the nature of the solutions will depend on whether $\sqrt{\alpha^2 - \omega_0^2} \lesseqqgtr 0$. Thus, near the stable equilibria, depending on the parameters, the phase curves will correspond to the underdamped, critical, or overdamped motions discussed above.

Near the unstable equilibria the behavior of the phase curves are determined by (22) and (24). These are equivalent to the second order differential equation of motion $\ddot{\theta} + 2\alpha\dot{\theta} - \omega_0^2(\theta - n_o\pi) = 0$. Now, let $\lambda_o = \theta - n_o\pi$. The equation of motion then becomes: $\ddot{\lambda}_o + 2\alpha\dot{\lambda}_o - \omega_0^2\lambda_o = 0$. Again, for a solution of the form $\lambda_o = e^{rt}$ to hold, the characteristic equation must satisfy $r^2 + 2\alpha r - \omega_0^2 = 0$. The two roots of this quadratic equation are then $r_1 = -\alpha + \sqrt{\alpha^2 + \omega_0^2}$ and $r_2 = -\alpha - \sqrt{\alpha^2 + \omega_0^2}$. Since the discriminant $\alpha^2 + \omega_0^2$ is always positive and that $\sqrt{\alpha^2 + \omega_0^2} > \alpha$, we infer that $r_1 > 0$ and $r_2 < 0$. Therefore, let $r_1 = u_1$ and $r_2 = -u_2$ where $u_1 > 0$ and $u_2 > 0$. The solution for $\lambda_o = \theta - n_o\pi$ then can be written in the form $Be^{u_1 t} + Ce^{-u_2 t}$ where B and C are constants to be determined via initial conditions. Therefore, the solutions near an unstable critical point take the form,

$$\theta = Be^{u_1 t} + Ce^{-u_2 t} + n_o\pi \tag{25}$$

$$\dot{\theta} = Bu_1 e^{u_1 t} - Cu_2 e^{u_2 t} \tag{26}$$

If the initial conditions near an unstable equilibrium point are θ_0 and $\dot{\theta}_0$, then, using the two equations above with $t = 0$, we find the constants to be

$$B = \frac{\dot{\theta}_0 + u_2(\theta_0 - n_o\pi)}{u_1 + u_2}, \quad C = \frac{-\dot{\theta}_0 + u_1(\theta_0 - n_o\pi)}{u_1 + u_2}.$$

The solution curves near the unstable equilibrium point π determined by the equations (25) and (26) are shown in Figure 12.

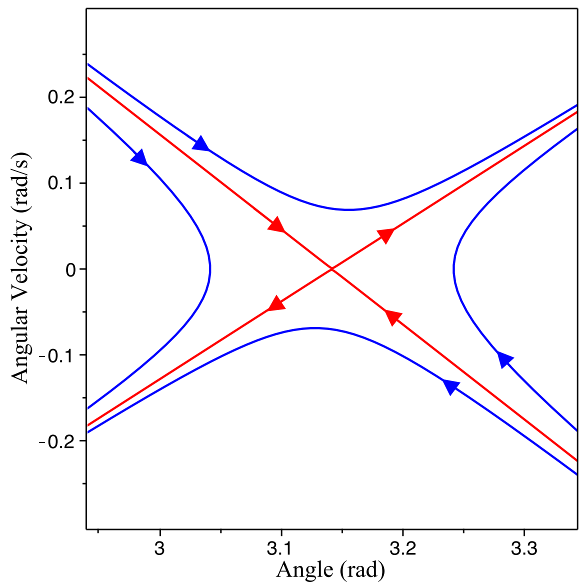

Figure 12: Phase curves for the damped pendulum near the unstable critical point π determined via the linear approximation equations (25) and (26). The asymptotes (red) have been created with the initial $(t = 0)$ conditions $\theta_0 = \pi$ and $\dot{\theta}_0 = \pm 0.001$. The left and right branches (blue) have been created with the initial conditions $\theta_0 = \pi \pm 0.1$ and $\dot{\theta}_0 = 0$; The top and bottom branches (blue) have been created with the initial conditions $\theta_0 = \pi$ and $\dot{\theta}_0 = \pm 0.07$. For all these segments time runs to ± 10 s from $t = 0$. The parameters were set at $\alpha = 0.1$ s^{-1} and $\omega_0 = 1$ s^{-1}.

♣