The Parabolic Pendulum

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This essay concerns the parabolic pendulum where we consider a particle of mass m slide back and forth without friction along a parabolic wireframe. We will first derive the equation of motion for the particle constrained to move along a general planar function $y = f(x)$ and then specialize it to the parabola; we will then derive the expression for the period for the parabolic pendulum.

Let the zero potential energy level of the particle be the x-axis and let the coordinates of the particle at a general point be (x, y). Since the particle is constrained to move along the differentiable function $y = f(x)$, the number of degrees of freedom available to the particle is actually one [two coordinates (x, y) in the plane minus the single constraint $y = f(x)$]. The kinetic and the potential energies are then given by:

$$\mathcal{E}_k = \frac{1}{2}mv^2, \ \ \mathcal{E}_p = mgy,$$

where

$$v^2 = \dot{x}^2 + \dot{y}^2.$$

Now,

$$\dot{y} = \frac{dy}{dt} = \frac{df(x)}{dt} = \frac{df(x)}{dx} \cdot \frac{dx}{dt} = f'(x) \cdot \dot{x},$$

where $f'(x) = df(x)/dx$. Therefore,

$$\mathcal{E}_k = \frac{1}{2}m\dot{x}^2[1 + f'^2(x)], \ \ \mathcal{E}_p = mgf(x).$$

The Lagrangian, $\mathcal{L} = \mathcal{E}_k - \mathcal{E}_p$, is then

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2[1 + f'^2(x)] - mgf(x).$$

To obtain the equation of motion for the particle we consider the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \left(\frac{\partial \mathcal{L}}{\partial x}\right) = 0.$$

From the Lagrangian, we obtain that,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}[1 + f'^2(x)], \quad \frac{\partial \mathcal{L}}{\partial x} = mf'(x)[\dot{x}^2 f''(x) - g],$$

where $f''(x) = df'(x)/dx$.

Thus,

$$\begin{aligned}
\frac{d}{dt}\left(m\dot{x}[1 + f'^2(x)]\right) - mf'(x)[\dot{x}^2 f''(x) - g] &= 0, \\
[1 + f'^2(x)]\ddot{x} + \dot{x} \cdot 2f'(x)\frac{d}{dt}f'(x) - f'(x)[\dot{x}^2 f''(x) - g] &= 0, \\
[1 + f'^2(x)]\ddot{x} + \dot{x} \cdot 2f'(x)\frac{df'(x)}{dx} \cdot \frac{dx}{dt} - f'(x)[\dot{x}^2 f''(x) - g] &= 0, \\
[1 + f'^2(x)]\ddot{x} + 2\dot{x}^2 f'(x)f''(x) - \dot{x}^2 f'(x)f''(x) + gf'(x) &= 0,
\end{aligned}$$

leading to the equation of motion,

$$\boxed{[1 + f'^2(x)]\ddot{x} + \dot{x}^2 f'(x)f''(x) + gf'(x) = 0.} \tag{1}$$

We do not expect any acceleration along the x-axis if the particle is moving parallel to that axis. This is because the only force acting on the particle would be gravity, which acts vertically downward resulting in a vanishing horizontal force component. That is, for $f'(x) = 0$ we expect $\ddot{x} = 0$, which the above equation of motion readily confirms.

Given that x is the generalized coordinate of the system, let us compute the generalized momentum p_x, which is given by

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}[1 + f'^2(x)]. \tag{2}$$

Thus, the Hamiltonian ($\mathcal{H}$) of the system is,

$$\mathcal{H} = p_x\dot{x} - \mathcal{L} = m\dot{x}^2[1 + f'^2(x)] - \left(\frac{1}{2}m\dot{x}^2[1 + f'^2(x)] - mgf(x)\right) = \frac{1}{2}m\dot{x}^2[1 + f'^2(x)] + mgf(x)$$

Since $\mathcal{H}$ does not explicitly depend on time t, it is conserved. Noting that the total energy $\mathcal{E} = \mathcal{E}_k + \mathcal{E}_p = \mathcal{H}$, we conclude that the energy of the system is also conserved. Using the generalized momentum, we can re-express $\mathcal{H}$ and $\mathcal{E}$ as,

$$\boxed{\mathcal{H} = \mathcal{E} = \frac{p_x^2}{2m[1 + f'^2(x)]} + mgf(x), \quad p_x = m\dot{x}[1 + f'^2(x)]} \tag{3}$$

If the particle is released from rest at $x = x_0$, then $p_{x_0} = 0$ at $x = x_0$. The total energy $\mathcal{E}$, which is conserved, is then given by $\mathcal{E} = mgf(x_0)$. Therefore,

$$p_x = \sqrt{2m^2 g[1 + f'^2(x)][f(x_0) - f(x)]}.$$

Using (2), we then have,

$$\dot{x} = \frac{dx}{dt} = \frac{p_x}{m[1 + f'^2(x)]} = \frac{\sqrt{2m^2 g[1 + f'^2(x)][f(x_0) - f(x)]}}{m[1 + f'^2(x)]} = \sqrt{\frac{2g[f(x_0) - f(x)]}{1 + f'^2(x)}}. \tag{4}$$

We can now specialize (4) to the case of the parabolic pendulum to find its period. Let us assume that the particle is now constrained to move on a parabola of the form $y = f(x) = ax^2$, where $a > 0$. Then, $f'(x) = 2ax$. (a has dimensions of inverse length; $a > 0$ guarantees that the parabola has a minimum at $x = 0$, thereby allowing the particle to librate about the minimum of the parabola under gravity.) Two parabolas with $a = 1$ and $a = 2$ are shown in Figure 1.

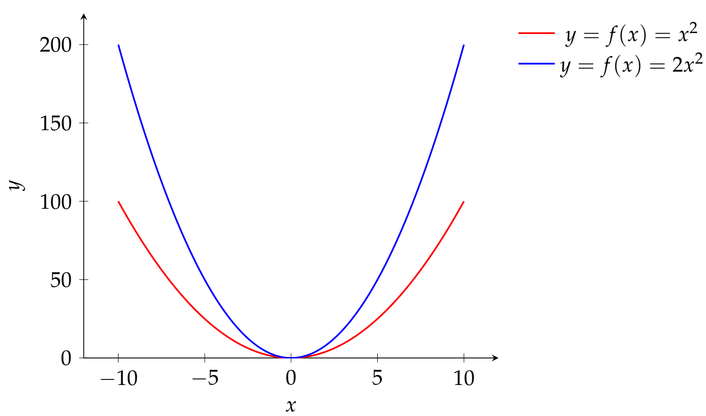

Figure 1: Two parabolas of the form $y = f(x) = ax^2$ with $a = 1$ (red) and $a = 2$ (blue).

Given the symmetry of this parabola about the y-axis along with that of energy conservation, when released from $+x_0$ ($\neq 0$), it will come to instantaneous rest at $-x_0$ and will move back to $+x_0$; the motion will then repeat indefinitely. Thus, using (4), the period (T_{pp}) of the parabolic pendulum is given by,

$$T_{pp} = \int dt = 2\int_{-x_0}^{x_0} \sqrt{\frac{1+f'^2(x)}{2g[f(x_0)-f(x)]}}\, dx = \sqrt{\frac{2}{ga}} \int_{-x_0}^{x_0} \sqrt{\frac{1+4a^2x^2}{x_0^2-x^2}}\, dx \tag{5}$$

If we choose $x = x_0 \sin\theta$, then $dx = x_0 \cos\theta \cdot d\theta$, and the new limits will run from $[-\pi/2, \pi/2]$. The above integral then reduces to,

$$T_{pp} = \sqrt{\frac{2}{ga}} \int_{-\pi/2}^{\pi/2} \sqrt{1+4a^2x_0^2\sin^2\theta}\, d\theta.$$

Since the integrand in the above integral is an even function in θ, it can be expressed with limits running from $[0, \pi/2]$ with an overall multiplicative factor of two. This finally leads to

$$\boxed{T_{pp} = \sqrt{\frac{8}{ga}} \int_{0}^{\pi/2} \sqrt{1+4a^2x_0^2\sin^2\theta}\, d\theta,\ x_0 \neq 0,\ a > 0.} \tag{6}$$

Figure 2 shows the variation in the period of the parabolic pendulum as the parameter a of the parabola is varied for a fixed starting point of the particle. Similarly, Figure 3 shows the variation in the period as the starting point is varied for a fixed a. Figure 4 shows the phase portraits of the parabolic pendulum for the parameter values $a = 2\ \mathrm{m}^{-1}$ and $a = 4\ \mathrm{m}^{-1}$.

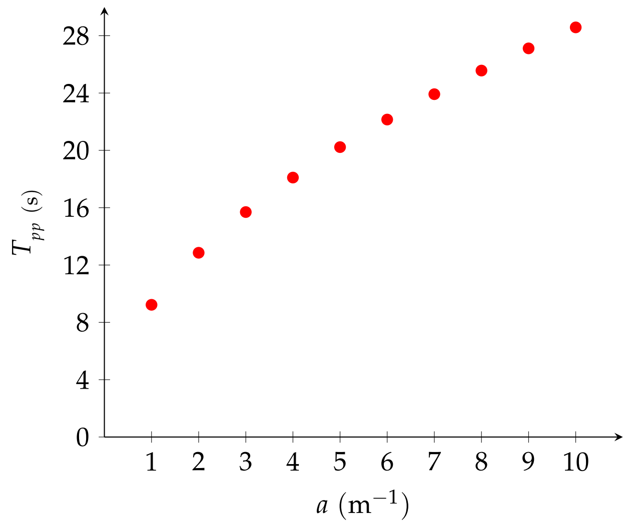

Figure 2: The period (T_{pp}) of the parabolic pendulum of the form $y = f(x) = ax^2$ as a function of the parameter a (> 0) of the parabola when the particle is released from rest at $x_0 = 5$ m. The periods are computed according to (6) with $g = 9.8\ \mathrm{ms}^{-2}$.

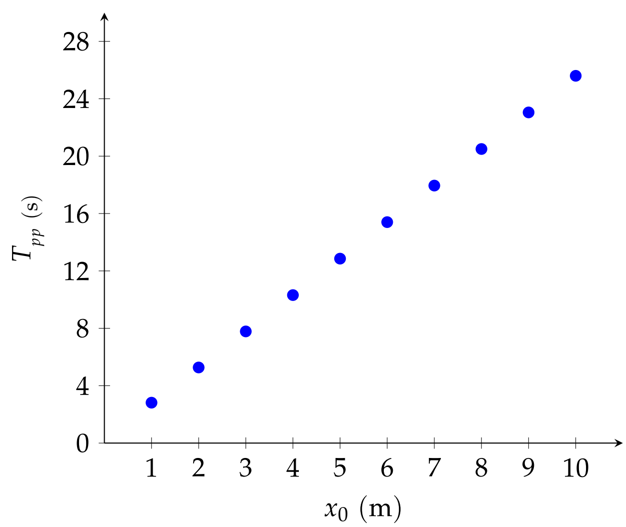

Figure 3: The period (T_{pp}) of the parabolic pendulum $y = f(x) = 2x^2$ (that is, when $a = 2\ \mathrm{m}^{-1}$) as a function of starting values x_0. The particle is released from rest in all cases. The periods are computed according to (6) with $g = 9.8\ \mathrm{ms}^{-2}$. The period scales linearly with the starting position.

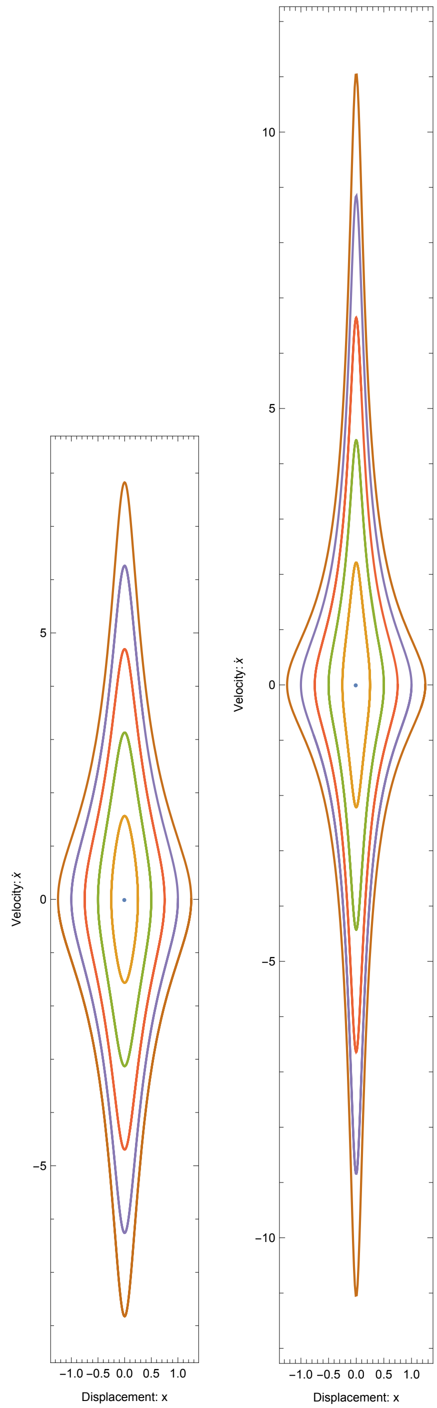

Figure 4: Phase portraits of the parabolic pendulum in the phase space given by $(x, \dot{x})$ for two values of the parameter a: $a = 2\ \mathrm{m}^{-1}$ (left) and $a = 4\ \mathrm{m}^{-1}$ (right) with $g = 9.8\ \mathrm{ms}^{-2}$. The particle thus librates along the function $y = f(x) = 2x^2$ on the left and along the function $y = f(x) = 4x^2$ on the right. The displacement is in m and the velocity is in ms^{-1}. The phase curves were obtained by solving the equation of motion given in (1). For the parabolic pendulum where $y = f(x) = ax^2$, this equation of motion takes the form: $(1 + 4a^2x^2)\ddot{x} + 2a(2a\dot{x}^2 + g)x = 0$.

Instead of the phase space $(x, \dot{x})$ we can utilize the phase space (x, p_x). The phase curves for a particle constrained to librate along the function $y = f(x) = x^2$ is shown in Figure 5. Here we see that if the particle is released from rest very close to the origin, then the closed phase curve has an elliptical shape. This shape becomes flatter if the particle is released from rest further away from the origin and it may then demonstrate a dip in p_x at the origin. This is due to the fact that the generalized momentum (p_x) is an interplay between the velocity ($\dot{x}$) and the slope ($f'(x) = 2x$) of the parabola as governed by the expression: $p_x = m\dot{x}[1 + f'^2(x)] = m\dot{x}(1 + 4x^2)$. When the particle is released from rest further out from the origin, it achieves a higher velocity (relative to when it is released closer to the origin) as it approaches the origin; however, the slope of the parabola also decreases as the particle approaches the origin – vanishing at the origin – and thereby leaving only the velocity to contribute to the generalized momentum (apart from the constant mass of the particle, of course).

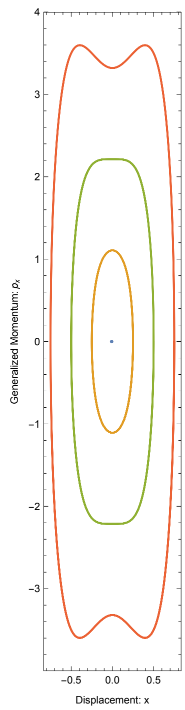

Figure 5: Phase portrait of the parabolic pendulum with $a = 1\ \mathrm{m}^{-1}$ in the phase space given by (x, p_x). The particle is thus constrained to librate along the function $y = f(x) = x^2$. Phase curves become flatter the further out from the origin the particle is released from rest along the parabola and they can demonstrate a dip in p_x at the origin. This is due to the fact that the generalized momentum (p_x) is an interplay between the velocity ($\dot{x}$) and the slope ($f'(x) = 2x$) of the parabola as governed by the expression: $p_x = m\dot{x}[1 + f'^2(x)]$.

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