Galileo's Law of the Pendulum and an Unsolved Math Problem

Rasil Warnakulasooriya

The story goes that Galileo discovered the law of the pendulum – that the period of a pendulum executing *small* swings is independent of the angle at which it is released – while timing the swings of a chandelier in a cathedral in Pisa, Italy, using his pulse. While this story may not be historically accurate – the real stories are always more complex and nuanced – Galileo did time the motion of the pendulums using water clocks and measurement sticks to a considerable degree of accuracy in later experiments*. In so doing, he established the connection between free fall and the pendulum swings, thereby arriving at the time-squared law of motion: that the distance an object falls vertically from rest is proportional to square of the time elapsed. Galileo arrived at the time-squared law of motion for a falling object by noting the sequence of odd numbers (1, 3, 5, 7, 9, . . .) associated with such motion. We thus immediately have a beautiful connection between motion and numbers.

Where this sequence of odd numbers come from can be discerned using the kinematical formula that gives the distance (d) as a function of time (t) for an object dropped vertically from rest: $d = (1/2)gt^2$, which Galileo did not know nor derive; this formula had to wait until Newton's formulation of the laws of motion and their expression via calculus. The formula applies when air friction and other disturbances to the motion is minimal so that they can be ignored for all practical purposes. Galileo did not even know about the gravitational constant g, so his approach to arriving at the time-squared law was indirect†. Consider the ratio $2d/g = d'$, which has units of time-squared. We thus have the relation $d' = t^2$. For the time sequence $t = 0, 1, 2, 3, 4, 5, \ldots$, we thus get the sequence $d' = 0, 1, 4, 9, 16, 25, \ldots$. The sequence of the difference between successive numbers of d' is then: $1, 3, 5, 7, 9, \ldots$. Galileo was able to infer the time-squared law of motion from this odd-numbered sequence since the law underpins the resulting sequence. In other words, during the first second of motion the object moves a "distance" (measured in time-squared units) of 1 unit; during the next second of motion the object moves an additional "distance" of 3 units, and so on. Since Galileo did not even utilize algebra as we do so today, his computations were very much in the tradition of the ancient Greek mathematicians whose collective work was given the logical structure by Euclid, which has become the hallmark mathematics ever since. As such, Galileo's main tools were geometry, whole numbers, and their ratios. We now turn to a great discovery of motion by Galileo, which is fundamentally geometric in character. This discovery then leads us to understand how it may have lead Galileo to discover the law of the pendulum stated above. Assuming frictionless motion, we first state the discovery:

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*The unit of length Galileo used was the *punto*, which is equivalent to 0.94 mm. His unit of time was the *tempo*, which is equivalent to 16 grains of flow or 1/92 sec (1 ounce being 480 grains).

†Galileo was actually tracking the overall speed of the object and did not arrive at the time-squared law using the distances directly. For details on Galileo's experiments, his experimental methods, and computations, the reader should reference *Galileo: Pioneer Scientist* by Stillman Drake, University of Toronto Press, 1990.

Galileo's discovery

Consider the vertical diameter and a non-diametrical chord of a circle such that the two share a common point at the bottom of the circle. Then, the time it takes a particle to fall from rest along the full length of the vertical diameter is identical to the time it takes the particle to fall from rest along the full length of the non-diametrical chord of the circle.

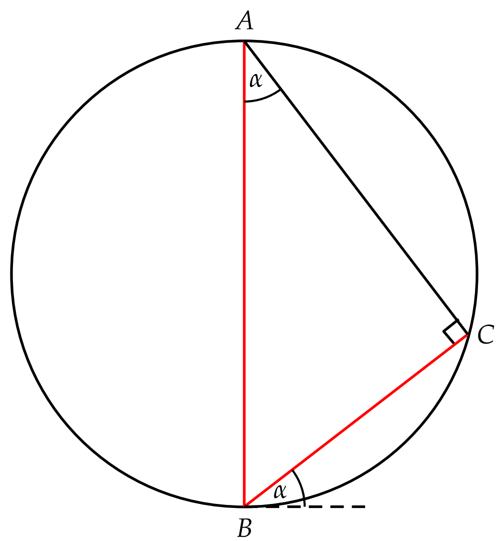

Figure 1: The time it takes a particle to fall vertically from rest along the diameter AB of the circle is the same as the time it takes it to fall along the non-diametrical chord CB. Chord CB is at an angle α to the horizontal. | Drawing by AD.

To see the truth in this statement, consider Figure 1. AB is the vertical diameter and CB is the non-diametrical chord of the circle, both meeting at the bottom of the circle at B. We deduce from Euclidean geometry that $\angle\, ACB = 90^\circ$. Therefore, if the chord is at an angle α to the horizontal, then $\angle\, CAB = \alpha$. Therefore, $AB \cdot \sin\alpha = CB$. Considering the kinematics, if the particle is released from rest at A, then $AB = (1/2)g \cdot t_{AB}^2$, where t_{AB} is the time it takes the particle to traverse the distance AB. If the particle is released at C, the gravitational acceleration acting along the chord CB is $g \sin\alpha$. Thus, $CB = (1/2)g\sin\alpha \cdot t_{CB}^2$, where t_{CB} is the time it takes the particle to traverse the distance CB. Thus,

$$\boxed{\frac{t_{AB}^2}{t_{CB}^2} = \frac{AB \cdot \sin\alpha}{CB} = \frac{CB}{CB} = 1 \implies t_{AB} = t_{CB}.} \tag{1}$$

The validity of the above statement is hence proved. Now, there is nothing special about the chosen non-diametrical chord. As such, regardless of the angle α of the chord to the horizontal, the time it takes for a particle to fall along the full length of the chord to the bottom of the circle is the same as the time it takes for the particle to fall along its vertical diameter. The latter time is only determined by the length of the diameter, which is fixed for a given circle. Therefore, the time it takes for a particle to fall along the full length of a non-diametrical chord is independent of the length of the chord! As α increases, the chord length increases, but since $\sin \alpha$ also increases, the increase in length is compensated by the increase in the gravitational acceleration component $g \sin \alpha$ along the chord, thereby leaving the travel time along the chord unaffected.

Having arrived at this result of time invariance for the chords of the circle, let us now consider a small arc along the perimeter of the circle such that one end of the arc is at B. If the arc is small enough to be approximated well by a chord of the circle having the same end points as the arc (one end point being B), then the time it takes a particle to move along the arc from rest to the bottom of the circle is the same as the time it takes the particle to fall along the vertical diameter of the circle. If we select a different arc, which also can be approximated well by a chord of the circle, then this means, that the time it takes for the particle to move along this new arc from rest to the bottom of the circle is also the same as the time it takes for it to fall along the vertical diameter of the circle. Since the time it takes the particle to fall along the vertical diameter is fixed, we therefore can deduce that the time it takes for the particle to move along the two different arcs is independent of the arc as long as these arcs can be approximated well by the chords of the circle. If these small arcs are generated by the small swings of a simple pendulum pivoted at the center of the circle (the particle in that case being the pendulum bob), then we arrive at Galileo's discovery that the period of the pendulum for small swings is independent of the release angle (also known as the amplitude).

Galileo's law of the pendulum

The period of the simple pendulum is independent of the release angle for small swings.

For completeness, let us look at how much the arc length (l_a) of the circle differs from its chord length (l_c) as a function of the release angle (θ) of the pendulum. For this purpose consider Figure 2.

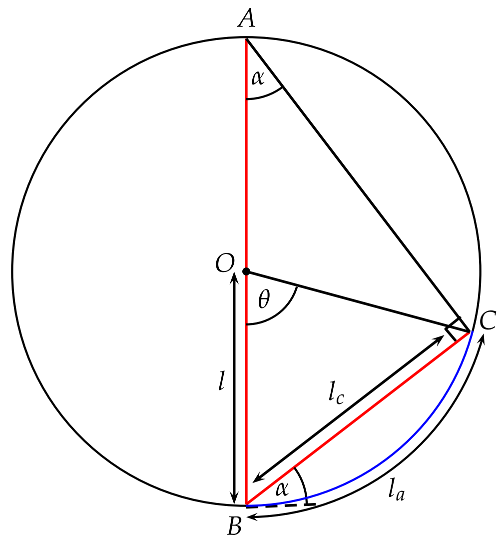

Figure 2: The pendulum of length l, which defines the radius of the circle, is pivoted at point O, the center of the circle. The length of the chord CB is l_c and the arc length along the perimeter of the circle having the same end points as the chord is l_a; chord CB is at an angle α to the horizontal. It can be shown that $\frac{l_a}{l_c} = \frac{\theta/2}{\sin(\theta/2)}$. | Drawing by AD.

The pendulum is pivoted at O (the center of the circle) and has length l, which defines the radius of the circle. Since OA and OC are radii, $OA = OC = l$, and therefore, $\angle OAC = \angle OCA = \alpha$. Thus $\theta = 2\alpha$. Since $l_a = l\theta$, and $l_c = 2l \sin\alpha = 2l \sin(\theta/2)$,

$$\frac{l_a}{l_c} = \frac{\theta/2}{\sin(\theta/2)}.$$

This function is plotted in Figure 3. It is immediately clear that the arc length is greater than the chord length for $\theta \neq 0$; however, there is a range of small release angles for which they are nearly identical, and therefore, for which, Galileo's law of the pendulum would hold. Although the arc length does not deviate by much in comparison to the chord length – even at $\theta = \pi/2$, l_a is less than 1.2 times that of l_c – this difference is significant enough to show appreciable difference in the period of the pendulum in comparison to the period computed according to small swings.

Before getting back to the pendulum, let us take this moment to reflect a little more on the ratio between the arc length and the chord length. Let us write the above ratio as,

$$\frac{l_c}{l_a} = \frac{\sin(\theta/2)}{\theta/2} = \frac{\sin\beta}{\beta}, \quad \beta = \frac{\theta}{2}.$$

In learning calculus, especially the concept of limits, we come across the result,

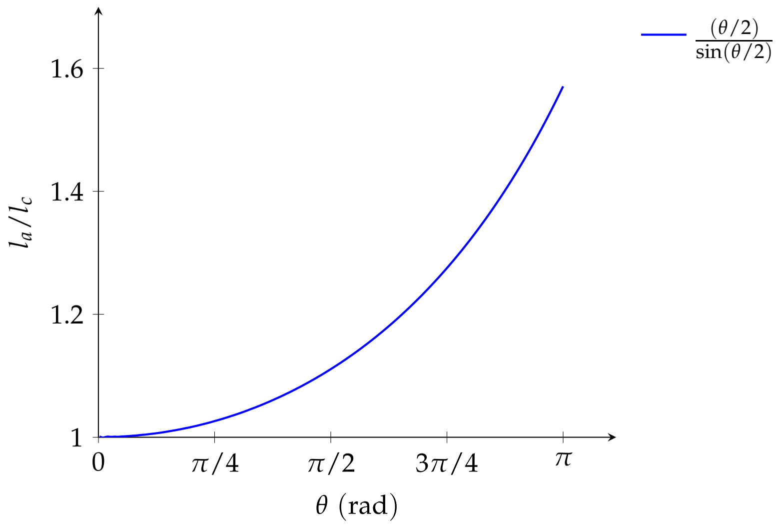

Figure 3: The ratio of the arc length (l_a) to that of the chord length (l_c) as a function of the release angle (θ) of the simple pendulum. The lengths are defined in the Figure 2. There is a range of small release angles for which the two lengths are nearly identical, and therefore, for which, Galileo's law of the pendulum holds.

$$\boxed{\lim_{\beta \to 0} \frac{\sin \beta}{\beta} = 1,} \tag{2}$$

though its meaning may often be hidden. Identifying the ratio of the sine of an angle to that of the angle itself as the ratio of chord length to that of the arc length, we can infer the meaning of the above limit being equal to one: it is that the chord length approaches that of the arc length as the angle β (and therefore, the release angle θ of the pendulum) goes to zero. In other words, consider a circle and an inscribed n-gon within the circle. Each side of the n-gon is a chord of the circle, and let the half angle between two radii that defines a side of the n-gon be β. The sum of the lengths of the n sides (chords) will give the perimeter of the n-gon. Now, as $n \to \infty$, the n-gon will approximate the circumference of the circle more and more accurately, and in the limit where the n-gon has infinite sides it will be the circle. So the ratio of the circumference of the circle to that of the perimeter of the n-gon in this limit will be exactly 1. But at this limit, the chord length of the n-gon will be zero, and hence, in the limit $n \to \infty$, $\beta \to 0$. Thus, in the limit $\beta \to 0$, $(\sin \beta)/\beta = 1$ since it is just a reflection of the ratio between the circumference of the circle to that of the perimeter of the infinite-side n-gon. This is the geometric meaning behind the above famous limit, which naturally arose in our attempt to understand Galileo's law of the pendulum!

Returning back to the pendulum, we now highlight a universal constant that connects the simple pendulum motion to free fall, the numerical value of which Galileo was able to arrive at experimentally. This universal constant is $\pi^2/8$, which is the ratio of the the distance a particle falls vertically from rest to that of the length of the pendulum such that the time it takes for the vertical fall is the same as the time it takes the simple pendulum to swing from its release position to the lowest point in its motion when released from rest at a small angle (measured relative to the downward vertical).

To arrive at this universal constant we then need the time it takes a simple pendulum to reach the lowest point of its motion when released from rest at a small angle. For this essay let us simply state the period (T) of the simple pendulum for small swings, which is

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}} \quad \text{for small swings.}} \tag{3}$$

We thus see that, for small swings, the period of the simple pendulum is independent of its release angle, reflecting Galileo's law of the pendulum. The period is only dependent on the length (l) of the pendulum and the gravitational constant (g). Given the dependency of the period on g, the same pendulum will have different periods depending on where it is located. For example, on the Moon, where the gravitational acceleration is about 1.6 ms^{-2}, which is about 16.6% that of the Earth's gravitational acceleration ($\sim$ 9.8 ms^{-2}), we expect the period for small swings of the pendulum to increase by a factor of about 2.4. Hence the period itself is location-dependent. Now, the time it takes the simple pendulum to reach the lowest point of its motion is simply one-fourth of the period. Thus the time of interest for the present discussion is

$$T_{1/4} = \frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{l}{g}} \quad \text{for small swings.}$$

The distance ($d_{T_{1/4}}$) a particle falls vertically from rest during this time interval ($\Delta t = T_{1/4}$) is then,

$$d_{T_{1/4}} = \frac{1}{2} g T_{1/4}^2 = \frac{\pi^2}{8} l \implies \frac{d_{T_{1/4}}}{l} = \frac{\pi^2}{8}.$$

Thus,

$$\boxed{\frac{\text{distance of vertical fall from rest}|_{\Delta\mathrm{t}=\mathrm{T}_{1/4}}}{\text{length of the simple pendulum}} = \frac{d_{T_{1/4}}}{l} = \frac{\pi^2}{8}.} \tag{4}$$

So the distance that an object falls vertically from rest in the time it takes the simple pendulum to swing to its lowest position (when released from rest at a small angle to the downward vertical) to that of the length of the pendulum is $\pi^2/8$. This number does not depend on the location of the pendulum, and therefore, holds true anywhere in the universe (assuming that other disturbances are absent, of course). $\pi^2/8$ is a special number. First, it is an irrational number, meaning, that it cannot be expressed (in this particular instance) as the ratio of two positive integers*. Thus the two lengths, $d_{T_{1/4}}$ and l, are not commensurable: that is, there is no common unit of length such that $d_{T_{1/4}}$ and l are whole multiples of the common unit of length. Second, this number results if one sums the infinite series of the squared reciprocals of positive odd numbers. (Again, the odd numbers make an appearance!) Given the inherent beauty of infinite series, we now turn to a derivation of this second fact before closing this essay.

Our first goal is to show that the square reciprocals of positive integers sum to $\pi^2/6$, which was demonstrated by Euler. The proof starts with the series expansion for $\sin x$ given by,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Dividing by $x (\neq 0)$, we obtain,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots .$$

We now consider the non-zero x values for which $(\sin x)/x = 0$. We immediately recognize that this equation is satisfied whenever $\sin x = 0$. The non-zero values of x that satisfies $\sin x = 0$ are: $x = \pm\pi,\ \pm 2\pi,\ \pm 3\pi,\ \pm 4\pi,\ \ldots$. Thus the zeros of $(\sin x)/x$ can be written in terms of the root expansion as,

$$\frac{\sin x}{x} = 0 = (x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)(x-4\pi)(x+4\pi)\cdots,$$

$$= 0 = \left[-\pi\left(1-\frac{x}{\pi}\right)\right]\left[\pi\left(1+\frac{x}{\pi}\right)\right]\left[-2\pi\left(1-\frac{x}{2\pi}\right)\right]\left[2\pi\left(1+\frac{x}{2\pi}\right)\right]\cdots,$$

$$= 0 = \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\left(1-\frac{x}{4\pi}\right)\left(1+\frac{x}{4\pi}\right)\cdots,$$

$$\frac{\sin x}{x} = 0 = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\cdots .$$

Expanding this last expression and collecting the terms up to x^2, we obtain

*A rational number is generally defined as the ratio a/b, where a, b are integers such that $b \neq 0$. For completeness, we note that numbers such as π, $\pi^2/6$, and $\pi^2/8$ are transcendental numbers, which means that such numbers do not arise as solutions to polynomials with integer coefficients.

$$\frac{\sin x}{x} = 1 - \frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)x^2 + \cdots = 0 = 1 - \frac{x^2}{3!} + \cdots .$$

Comparing the coefficients in front of the x^2 term, we conclude that

$$-\frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) = -\frac{1}{3!}.$$

Thus,

$$\boxed{S_1 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}} \tag{5}$$

Now consider the infinite sum of squared reciprocals of the positive odd integers:

$$S_2 = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Thus,

$$S_1 - S_2 = \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \cdots = \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) = \frac{1}{4}S_1.$$

This implies that,

$$\boxed{S_2 = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{3}{4}S_1 = \frac{\pi^2}{8}} \tag{6}$$

We have thus arrived at the result that $\pi^2/8$ is the infinite sum of the squared reciprocals of the positive odd integers, which is intimately connected with the motion of the simple pendulum as demonstrated above. Before closing, we wish to whet the reader's appetite by mentioning that although $\sum_{n\geq 1}(1/n^2)$ converges to the transcendental number $\pi^2/6$, it is not known whether $\sum_{n\geq 1}(1/n^3)$ similarly converges to a transcendental number. We can easily infer that the infinite sum of the reciprocals of the cubes of positive integers will converge to a number less than $\pi^2/6$; it is also proven with sophisticated arguments that the number to which the series would converge is irrational. But whether that irrational number is necessarily a transcendental number is an unsolved problem in mathematics at present.

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