The Probability of Finding the Pendulum Along a Swing

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Our interest in this essay concerns the probability of finding the simple pendulum between some angular range of its swing. For this let us first consider the small swings of a simple pendulum of length l. As is well known, the equation of motion for small swings of the simple pendulum, where angle θ is measured from the downward vertical, is given by

$$\ddot{\theta} = -\omega^2\theta, \tag{1}$$

where $\omega = \sqrt{g/l}$ and g is the constant gravitational acceleration. The negative sign in the above expression says that the angular acceleration is opposite to the angle measure (from the vertical) in the pendulum swing. That is, when the pendulum is in the right half of its swing, the angle θ is measured as positive, but the force component (with magnitude $mg\sin\theta$) tangent to the pendulum mass's trajectory is pointing to the left; in contrast, when the pendulum is in the left half of its swing, the angle θ is measured as negative, but the force component tangent to the pendulum mass's trajectory is now pointing to the right. In short, there is always a restoring force which works against the increase in the angle.

We now consider the variation in the pendulum's angular speed. Intuitively, we realize that from the symmetry of the pendulum's swing, that the speed of the pendulum must be the same for two identical positions on the left- and right-halves. In such identical places, the pendulum is at a common height from where the potential energy is measured and thus have the same potential energies, which in turn implies that those positions have the same kinetic energies since the total energy is conserved. If kinetic energies are the same, the speeds must be the same at those points. Thus the angular speed ($\dot{\theta}$) must be independent of whether θ is positive (right half) or negative (left half). The simplest way to achieve this is to make $\dot{\theta}$ dependent on θ^2. We also know that the pendulum momentarily stops when it reaches the end points of its swing, which corresponds to its release angle $\pm\alpha$. So when $\theta = \pm\alpha$, $\dot{\theta} = 0$. Combining with θ^2 dependency, it seems that $\dot{\theta}$ should be formed using $(\alpha^2 - \theta^2)$. The units of $\dot{\theta}$ is radians per second, while $(\alpha^2 - \theta^2)$ has units of radians squared. Noting that ω, which provides a characteristic of the motion of the pendulum, and that it has units of per second, we can put together all of the above heuristic arguments to express the angular speed of small swings as,

$$\boxed{\dot{\theta} = \omega\sqrt{\alpha^2 - \theta^2},\ -\alpha \le \theta \le \alpha;\ \alpha \text{ small.}} \tag{2}$$

The above result can be more rigorously verified by using energy conservation. Using the energy argument, we can deduce that, in general, $\dot{\theta}^2 = 2(g/l)(\cos\theta - \cos\alpha) = 2\omega^2(\cos\theta - \cos\alpha)$. For small swings, we can write $\cos\theta \approx 1 - (\theta^2/2)$ and $\cos\alpha \approx 1 - (\alpha^2/2)$. Substitution followed by taking the square root of $\dot{\theta}^2$ then yields (2).

We now turn to the probability of finding the pendulum between two angles for small swings. From (2) we note that,

$$dt = \frac{d\theta}{\omega\sqrt{\alpha^2 - \theta^2}}.$$

We now consider the infinitesimal time dt the pendulum spends in the angular range $d\theta$ relative to the total time it takes for it to swing from $-\alpha$ to $+\alpha$. This is half of the period T_0 $(= 2\pi\sqrt{l/g})$ associated with the small swings. That is,

$$\frac{1}{2}T_0 = T_{0[1/2]} = \pi\sqrt{\frac{l}{g}} = \frac{\pi}{\omega}.$$

The ratio $dt/T_{0[1/2]}$ can then be interpreted as the infinitesimal probability, $dP_0(\theta)$, of finding the pendulum in the angular range $d\theta$:

$$dP_0(\theta) = dt/T_{0[1/2]} = \frac{d\theta}{\pi\sqrt{\alpha^2 - \theta^2}} = p_0(\theta)d\theta,$$

where

$$p_0(\theta) = \frac{1}{\pi\sqrt{\alpha^2 - \theta^2}}$$

is the small-swing probability density for the pendulum. Hence, the probability of finding the pendulum, for example, between the angles $\pm\theta$ (for small α and θ) is,

$$\boxed{P_0(\theta) = \int_{-\theta}^{\theta} p_0(\theta)d\theta = \frac{1}{\pi}\int_{-\theta}^{\theta}\frac{d\theta}{\sqrt{\alpha^2 - \theta^2}} = \frac{1}{\pi}\sin^{-1}\left(\frac{\theta}{\alpha}\right)\Bigg|_{-\theta}^{\theta},\ -\alpha \le \theta \le \alpha,\ \alpha\ \text{small.}} \quad (3)$$

Figures 1 and 2 show the probability density and probabilities, respectively, for specific initial angles. The above integral evaluates to 1 when integrated between $\pm\alpha$. This simply justifies the integral since we expect the pendulum to be certainly found somewhere between $\pm\alpha$ when released at an angle $\pm\alpha$. Additionally, we see from Figure 1 that the likelihood of finding the pendulum is higher when it reaches closer to the release angle compared to when it is at the bottom of its swing. This must be the case since at the bottom of its swing the pendulum is moving faster than when it reaches closer to the release angle on either side of the swing. So snapshots taken at random times will reveal that the pendulum is more likely to be found at places (along the swing) where it is moving slower than at places where it is moving faster.

We can follow a similar line of argument to derive the probability of finding the pendulum when not constrained to small angles. For this purpose we start with the general expression $\dot{\theta}^2 = 2(g/l)(\cos\theta - \cos\alpha) = 2\omega^2(\cos\theta - \cos\alpha)$, which results from the energy considerations.

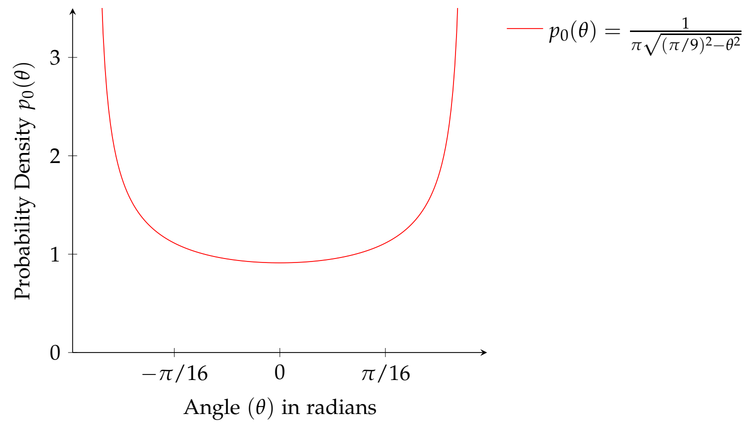

Figure 1: The probability density $p_0(\theta)$ against the angle (θ) in radians for a pendulum with release angle $\alpha = 20^\circ$ ($\frac{\pi}{9}$ radians).

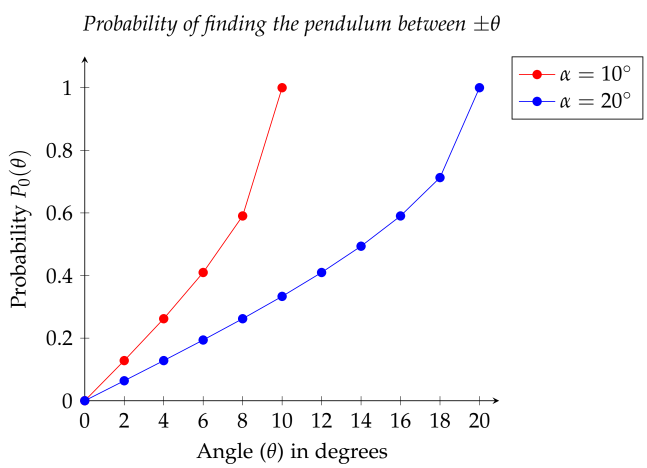

Figure 2: The probability $P_0(\theta)$ that the pendulum can be found between the angles $\pm\theta$ to the vertical, plotted for two initial angles ($\alpha = 10^\circ, 20^\circ$). For example, in the case where the release angle is $\alpha = 20^\circ$, the probability that the pendulum can be found between $\pm 12^\circ$ to the vertical is about 0.4. The pendulum is certain to be found between either side of the release angle ($\pm\alpha$), which is clearly the case, since in this case the associated probability is 1.

Taking the square root of this expression, we can rearrange it to be expressed for dt. Like before, we can then normalize it with respect to half the period of the swing, which in this case needs to be $\frac{1}{2}T_\alpha$, where T_α is given by (see the essay *The Period of a General Swing of the Simple Pendulum*)

$$T_\alpha = 2\pi\sqrt{\frac{l}{g}}\left\{1+\left(\frac{1}{2}\right)^2 k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6+\cdots\right\},\ k=\sin\frac{\alpha}{2}. \tag{4}$$

Thus the infinitesimal probability, $dP_\alpha(\theta)$, of finding the pendulum in the angular range $d\theta$ is given by,

$$dP_\alpha(\theta) = dt/T_{\alpha[1/2]} = \frac{d\theta}{\sqrt{2}\pi\left\{1+\sum_{n=1}^{\infty}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ \cdots\ \cdot\frac{2n-1}{2n}\right)^2 k^{2n}\right\}\sqrt{\cos\theta-\cos\alpha}} = p_\alpha(\theta)d\theta.$$

Denoting the infinite series in the expression as S_α, the general probability density, $p_\alpha(\theta)$, can be written as,

$$p_\alpha(\theta) = \frac{1}{\sqrt{2}\pi\left\{1+S_\alpha\right\}\sqrt{\cos\theta-\cos\alpha}}.$$

Hence, the probability of finding the pendulum, for example, between the angles $\pm\theta$ is,

$$\boxed{P_\alpha(\theta) = \int_{-\theta}^{\theta} p_\alpha(\theta)d\theta = \frac{1}{\sqrt{2}\pi\left\{1+S_\alpha\right\}}\int_{-\theta}^{\theta}\frac{d\theta}{\sqrt{\cos\theta-\cos\alpha}},\ -\alpha\le\theta\le\alpha.} \tag{5}$$

Figure 3 shows the probability density against the angle θ when the pendulum is released from an angle of 60°. Figure 4 shows the comparison of the probabilities – with and without the small angle approximation – that the pendulum would be between $\pm\theta$ when released at 60°: that is, with respect to the evaluations (3) and (5).

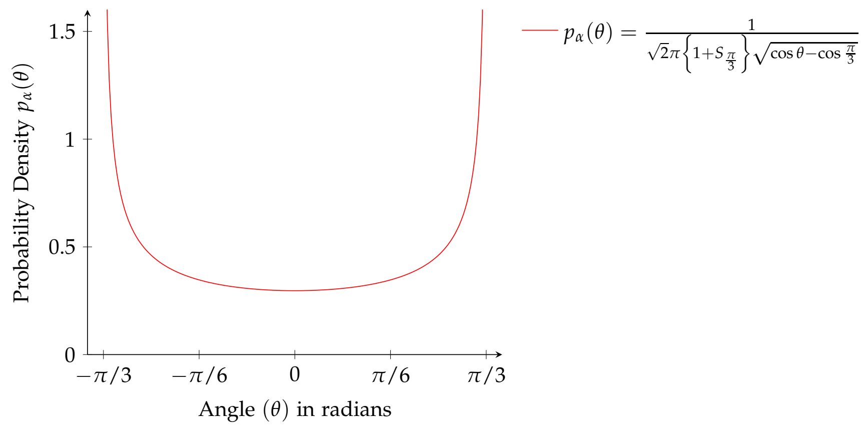

Figure 3: The probability density $p_\alpha(\theta)$ against the angle (θ) in radians for a pendulum with release angle $\alpha = 60^\circ$ ($\frac{\pi}{3}$ radians).

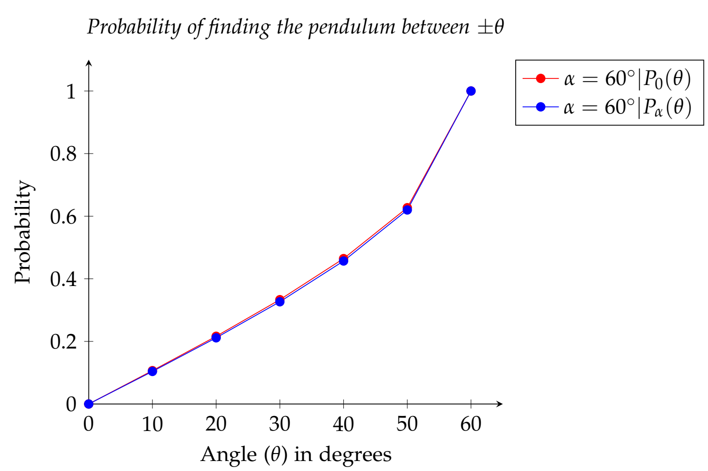

Figure 4: The probability of finding the pendulum between $\pm\theta$ with and without the small angle approximation when it is released at 60°. The probabilities are practically identical. $S_{\frac{\pi}{3}}$ is computed to one thousand terms when evaluating P_α.

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