Renormalizing the Pendulum: Computing the Period via the Arithmetic-Geometric Mean

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As is known, the general solution to the equation of motion of the simple pendulum, $\ddot{\theta} = -\omega^2 \sin\theta$, does not have a closed form. (Here, θ is the angle measured from the downward vertical and $\omega^2 = g/l$ where g is the constant gravitational acceleration and l is the length of the pendulum rod.) A closed form solution exists for small swings when $\sin\theta$ can be approximated by θ. In this essay we demonstrate how the period of the pendulum for a general swing that leads to librations can be determined using the notions of arithmetic and geometric means. This approach provides us with a very fast algorithm for computing the period, but most importantly, introduces us to some beautiful mathematics involving elliptic curves and elliptic integrals. Not surprisingly, we start with the kinetic and the potential energies of the pendulum, which are respectively given by,

$$\mathcal{E}_k = \frac{1}{2} m l^2 \dot{\theta}^2, \ \mathcal{E}_p = mgl(1 - \cos\theta),$$

where m is the mass of the pendulum bob, l is the length of the pendulum rod, and g is the gravitational acceleration (see Figure 1).

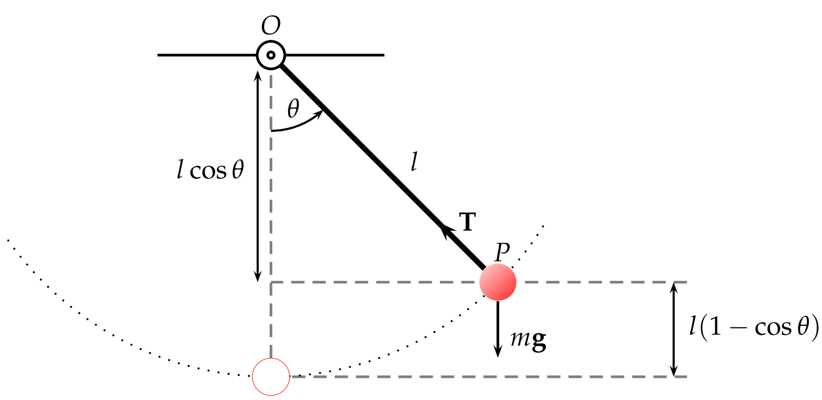

Figure 1: An instantaneous position (P) of the simple pendulum of length l with mass m pivoted at O. The angle measured from the downward vertical is θ. The potential energy of the pendulum is taken to be zero at the bottom of the swing. $\mathbf{T}$ and $m\mathbf{g}$ are the tension on the weightless rod and the weight of the bob, respectively. | Drawing by AD.

The sum of the energies, $\mathcal{E}_k + \mathcal{E}_p$ can then be written $ml^2[\frac{1}{2}\dot{\theta}^2 + \omega^2(1 - \cos\theta)]$, where $\omega^2 = g/l$. Thus, scaled by the constant factor ml^2, the total energy ($\mathcal{E}$) of the pendulum at any given instant can be written as

$$\boxed{\mathcal{E} = \frac{1}{2}\dot{\theta}^2 + \omega^2(1 - \cos\theta),} \tag{1}$$

where $\mathcal{E}$ has the same units as ω^2: per second squared (s^{-2}). The total energy is conserved since we consider frictionless motion. Let the pendulum be released from rest at an angle θ_0 so that $\dot{\theta}_0 = 0$. This angle determines the total energy $\mathcal{E}$, and the energy is highest when the pendulum is hanging vertically up at which $\theta_0 = \pi$. Thus $\mathcal{E}_{\text{max}} = \omega^2(1 - \cos\pi) = 2\omega^2$. Similarly, when the pendulum is released at $\theta_0 = 0$, it has the lowest total energy $\mathcal{E}_{\text{min}} = \omega^2(1 - \cos 0) = 0$. In both these cases the pendulum is in equilibrium, although this equilibrium is unstable when hanging up vertically, and is stable when hanging down vertically. Therefore, these two cases do not lead to librations or swings; hence, librations are only possible when $0 < \mathcal{E} < 2\omega^2$. Let us now define a dimensionless parameter k such that,

$$k^2 = \frac{\mathcal{E}}{2\omega^2}.$$

Therefore, $k = \pm 1$ when the pendulum is hanging up in unstable equilibrium, and $k = 0$ when it is hanging down in stable equilibrium. We can therefore consider $0 < k < 1$ which would correspond to librations or swinging motion of the simple pendulum. In addition, let us also define a variable x such that,

$$x = \frac{1}{k}\sin\frac{\theta}{2}.$$

We can now ask what the value of $x = x_0$ is when $\theta = \theta_0$, which corresponds to the moment of release, or in other words, at the points of maximum deflection of the pendulum during librations. For this we notice that since $1 - \cos\theta_0 = 2\sin^2(\theta_0/2)$, at release or at maximum deflection,

$$\mathcal{E} = \omega^2(1 - \cos\theta_0) = \omega^2 \cdot 2\sin^2\frac{\theta_0}{2} \implies \left(\frac{2\omega^2}{\mathcal{E}}\right)\sin^2\frac{\theta_0}{2} = \frac{1}{k^2}\sin^2\frac{\theta_0}{2} = 1.$$

But $(1/k)\sin(\theta_0/2) = x_0$. Therefore, $x_0^2 = 1$, and hence, $x_0 = \pm 1$ at release or at maximum deflection. Therefore, x values take between $[-1, 1]$ for librations of the simple pendulum with $x = 0$ corresponding to the lowest point of the pendulum's motion where $\theta = 0$. Differentiating x with respect to time t, we obtain,

$$\dot{x} = \frac{\dot{\theta}}{2k}\cos\frac{\theta}{2}.$$

Squaring this and using the identity $\cos^2(\theta/2) = 1 - \sin^2(\theta/2)$ along with the definition for x, we obtain, $\dot{\theta}^2/2 = 2k^2\dot{x}^2/(1 - k^2x^2)$; also, $1 - \cos\theta = 2\sin^2(\theta/2) = 2k^2x^2$, and by definition, $\mathcal{E} = 2\omega^2k^2$. Substituting these terms in the expression for $\mathcal{E}$, we obtain

$$\boxed{\dot{x}^2 = \omega^2(1-x^2)(1-k^2x^2), \;\; 0<k<1, \;\; -1 \le x \le 1 \;\; \text{Elliptic curves for librations.}} \tag{2}$$

The above expression encapsulates the librations of the pendulum in the form of elliptic curves, which look like distorted circles (see Figure 2); the elliptic curves, however, are not ellipses.

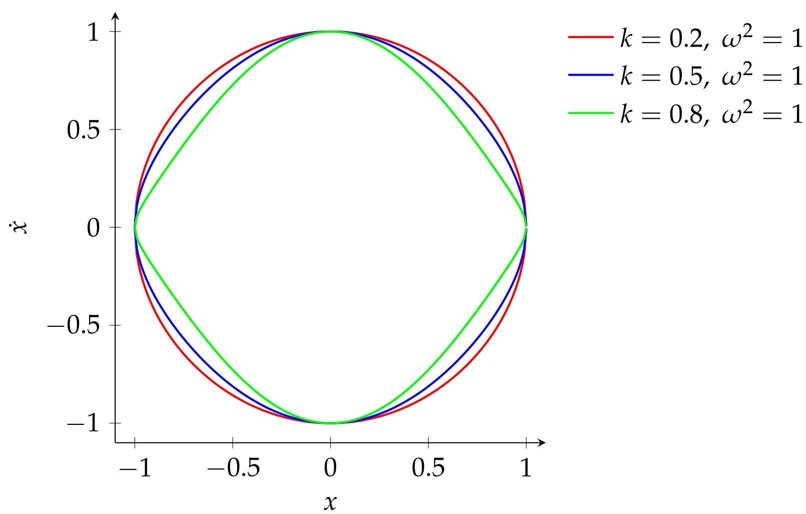

Figure 2: Elliptic curves corresponding to the librations of the pendulum as defined by the expression (2) for three different k values with $\omega^2 = 1$. The closer the k value is to 0 (with $\omega^2 = 1$) the more the corresponding elliptic curve would approximate a circle of unit radius. Similarly, the closer the k value is to 1 (with $\omega^2 = 1$) the more the corresponding elliptic curve would approximate a parabola. However, $k = 0$ or $k = 1$ would not correspond to the librations of the pendulum.

From (2),

$$dt = \frac{1}{\omega}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

The period (T) of the pendulum can therefore be written as,

$$\boxed{T = 4\int_{\text{release point}}^{\text{lowest point}} dt = \frac{4}{\omega}\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.} \tag{3}$$

The integral in the above expression is known as the complete elliptic integral (I_{ce}), which is a function of k; hence,

$$\boxed{I_{ce}(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.} \tag{4}$$

To find the period of the simple pendulum, then, we need to evaluate I_{ce}. The amazing fact is, that although we cannot evaluate this integral in closed form, it can be evaluated very quickly to a high degree of accuracy using successive iterations of the arithmetic and geometric means of two numbers which we will discuss below.

Let us first make the variable change $x = \sin\phi$; hence, $dx = \cos\phi \cdot d\phi$. The limits $[0, 1]$ of x corresponds to the limits $[0, \pi/2]$ of ϕ. Substitution in (4) yields,

$$I_{ce}(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}. \tag{5}$$

Therefore, (3) can be written as,

$$\begin{aligned} T &= \frac{4}{\omega}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \\ &= 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{\omega^2-\omega^2k^2\sin^2\phi}}, \\ &= 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{\omega^2(\cos^2\phi+\sin^2\phi)-\omega^2k^2\sin^2\phi}}, \\ &= 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{\omega^2\cos^2\phi+\omega^2(1-k^2)\sin^2\phi}}. \end{aligned}$$

We now define a new angular frequency $\Omega > 0$ such that $\Omega^2 = \omega^2(1-k^2)$. Therefore,

$$T = 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{\omega^2\cos^2\phi+\Omega^2\sin^2\phi)}}. \tag{6}$$

Before proceeding, let us delve a little on the nature of Ω. Given that we defined $k^2 = \mathcal{E}/(2\omega^2)$, it is the case that $1-k^2 = 1-\mathcal{E}/(2\omega^2)$. Thus,

$$\omega^2(1-k^2) = \omega^2 - \frac{\mathcal{E}}{2} = \Omega^2.$$

Therefore,

$$\omega^2 - \Omega^2 = (\omega - \Omega)(\omega + \Omega) = \frac{\mathcal{E}}{2}. \tag{7}$$

Now, clearly, $\omega > 0$ since it is determined by the finite positive values of the gravitational acceleration g and the pendulum length l. Since $\mathcal{E} > 0$ for librations (though $\mathcal{E}$ also has to be less than $2\omega^2$), it cannot be the case that $\Omega < 0$, since per (7) that could lead to scenarios where $\mathcal{E} < 0$, which is impossible. Ω cannot be 0 either, since by definition it is equal to $\omega^2(1-k^2)$, which is positive since k can never equal 1 for librations. This leaves us with the conclusion that $\Omega > 0$. Thus, for librations: $\omega > 0$, $\Omega > 0$, and $\mathcal{E} > 0$. According to (7), if $\mathcal{E} > 0$, and both ω and Ω are positive, then there is the further implication that $\omega > \Omega$. We collect these important conditions below:

$$\boxed{\omega > 0,\ \Omega > 0,\ \omega > \Omega.}$$

We now initiate a new transformation: $y = \Omega \tan\phi$; hence, $dy = \Omega(1 + \tan^2\phi) \cdot d\phi = \frac{1}{\Omega}(\Omega^2 + y^2) \cdot d\phi$. The limits $[0, \pi/2]$ of ϕ corresponds to the limits $[0, \infty)$ of y. Substitution in (6) now yields,

$$\begin{aligned} T &= 4\int_0^\infty \frac{dy}{\frac{1}{\Omega}(\Omega^2 + y^2)\sqrt{\omega^2\cos^2\phi + \Omega^2\sin^2\phi}}, \\ &= 4\int_0^\infty \frac{dy}{\frac{1}{\Omega}(\Omega^2 + y^2)\cos\phi\sqrt{\omega^2 + \Omega^2\tan^2\phi}}, \\ &= 4\int_0^\infty \frac{dy}{\frac{1}{\Omega}(\Omega^2 + y^2)\cos\phi\sqrt{\omega^2 + y^2}}. \end{aligned}$$

Now, since $\tan\phi = \sin\phi/\cos\phi$, $\sin^2\phi = \cos^2\phi\tan^2\phi$. But $\sin^2\phi + \cos^2\phi = 1$. Hence, $\cos^2\phi(1 + \tan^2\phi) = 1 \implies \cos^2\phi(1 + y^2/\Omega^2) = 1$. Thus, $\cos^2\phi = \Omega^2/(\Omega^2 + y^2)$, and therefore, $\cos\phi = \Omega/(\sqrt{\Omega^2 + y^2})$. Substituting this result for $\cos\phi$ in the last expression, we arrive at,

$$T = 4\int_0^\infty \frac{dy}{\sqrt{(\Omega^2 + y^2)(\omega^2 + y^2)}},$$

which is beautifully symmetric in $\Omega \leftrightarrow \omega$. Since the integrand is an even function of y, the above can be re-written as

$$\boxed{T = 2\int_{-\infty}^{\infty} \frac{dy}{\sqrt{(\Omega^2 + y^2)(\omega^2 + y^2)}}} \tag{8}$$

We now wish to reveal the hidden gem within this integral. It is that, if we let,

$$\omega \mapsto \omega' = \frac{\omega + \Omega}{2}, \ \Omega \mapsto \Omega' = \sqrt{\omega \cdot \Omega},$$

then,

$$\int_{-\infty}^{\infty} \frac{dy}{\sqrt{(\Omega^2 + y^2)(\omega^2 + y^2)}} = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{(\Omega'^2 + y^2)(\omega'^2 + y^2)}}$$

That is, if ω and Ω are replaced by the arithmetic mean $AM(\omega, \Omega) = (\omega + \Omega)/2$ and the geometric mean $GM(\omega, \Omega) = \sqrt{\omega \cdot \Omega}$, respectively, then the integral in (8) remains invariant. Thus,

$$\boxed{T = 2\int_{-\infty}^{\infty} \frac{dy}{\sqrt{(\Omega^2 + y^2)(\omega^2 + y^2)}} = 2\int_{-\infty}^{\infty} \frac{dy}{\sqrt{(GM(\omega, \Omega)^2 + y^2)(AM(\omega, \Omega)^2 + y^2)}}} \tag{9}$$

This is the secret to finding the period of the pendulum without attempting to integrate the elliptic integral, for which a closed form solution does not exist anyway. Let us first see the consequences of the above invariance and how it enables the evaluation of the period of the pendulum. We will then provide a proof of the invariance of (9).

Let us consider the first iteration from ω to $\omega' = (\omega + \Omega)/2$. We have established previously that $\omega > \Omega$. Thus, $\omega + \Omega > 2\Omega$, which implies that $(\omega + \Omega)/2 > \Omega$, and hence, $\omega' > \Omega$. Since $\Omega = 2\omega' - \omega$, it follows that $\omega' > 2\omega' - \omega$, and hence, $\omega > \omega'$. We therefore conclude that after the first iteration, $AM(\omega, \Omega) = \omega'$ is less than ω.

Similarly, let us consider the first iteration from Ω to $\Omega' = \sqrt{\omega \cdot \Omega}$. Again, since $\omega > \Omega$, $\omega \cdot \Omega > \Omega^2$, which implies that $\sqrt{\omega \cdot \Omega} > \Omega$, and hence, $\Omega' > \Omega$. We therefore conclude that after the first iteration, $GM(\omega, \Omega) = \Omega'$ is greater than Ω.

So, although ω has decreased to $\omega' = AM(\omega, \Omega)$ and Ω has increased to $\Omega' = GM(\omega, \Omega)$ after the first iteration, according to (9) the integral remains invariant, which in turn implies that the period of the pendulum can be evaluated using (ω', Ω') just as it can by using (ω, Ω). Let us assume that we continue the iterations such that we obtain the sequences $\omega_0 (= \omega)$, $\omega_1 (= \omega')$, $\omega_2, \cdots$ and $\Omega_0 (= \Omega)$, $\Omega_1 (= \Omega')$, $\Omega_2, \cdots$. Thus, for any natural number n (including 0),

$$\omega_{n+1} = \frac{\omega_n + \Omega_n}{2} = AM(\omega_n, \Omega_n), \ \ \Omega_{n+1} = \sqrt{\omega_n \cdot \Omega_n} = GM(\omega_n, \Omega_n), \ \ n \in \mathbb{N}_0,$$

where $\mathbb{N}_0$ is the set of natural numbers that include 0. By the previous verifications and by induction we can conclude that $\omega_{n+1} < \omega_n$ and $\Omega_{n+1} > \Omega_n$. Therefore, successive iterations lead to a decreasing sequence of ω_n and an increasing sequence of Ω_n, both of which converge to the same value in the limit $n \to \infty$. To see that they converge to a common value, first note that if a and b are positive real numbers such that $a > b > 0$, then the arithmetic mean of these two numbers is greater than their geometric mean. That is, if $a > b > 0$, then it is the case that $a - b > 0$, which implies that $(a-b)^2 > 0$. Thus, $a^2 - 2ab + b^2 > 0$. Adding $4ab$ to both sides, we obtain $a^2 + 2ab + b^2 > 4ab$, which implies $((a+b)/2)^2 > ab$. Hence, $AM(a,b) = (a+b)/2 > \sqrt{ab} = GM(a,b)$. Therefore, for any n, the condition $AM(\omega_n, \Omega_n) = \omega_{n+1} > \Omega_{n+1} = GM(\omega_n, \Omega_n)$ is satisfied. Thus, $\omega_{n+1} - \Omega_{n+1} > 0$. Since $\Omega_{n+1} > \Omega_n$, we can further conclude that $\omega_{n+1} - \Omega_n > \omega_{n+1} - \Omega_{n+1}$. But $\omega_{n+1} = (\omega_n + \Omega_n)/2$. Therefore, $(\omega_n + \Omega_n)/2 - \Omega_n > \omega_{n+1} - \Omega_{n+1}$, which implies $(\omega_n - \Omega_n)/2 > \omega_{n+1} - \Omega_{n+1} > 0$. Therefore, by induction, we conclude,

$$0 < \omega_n - \Omega_n < \frac{\omega - \Omega}{2^n}.$$

In the limit $n \to \infty$, from the first half of the above inequalities we are led to conclude that, $\omega_\infty > \Omega_\infty$; from the second half of the above inequalities, however, we are led to conclude that, $\omega_\infty < \Omega_\infty$. This is a contradiction and hence both statements cannot be true. We have no choice, then, than to declare that $\omega_\infty = \Omega_\infty$. In other words,

$$\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} \Omega_n = AGM(\omega, \Omega),$$

where $AGM(\omega, \Omega)$ stands for the common limit known as the arithmetic-geometric mean.

Now let us look at (7) as we iterate. For any iteration n, (7) can be written as,

$$(\omega_n - \Omega_n)(\omega_n + \Omega_n) = \frac{\mathcal{E}_n}{2}.$$

As we iterate, ω_n and Ω_n converge to $AGM(\omega, \Omega)$ as demonstrated above. This means that the difference $(\omega_n - \Omega_n)$ become smaller and smaller with each successive iteration. As a result, with each iteration, the energy $\mathcal{E}_n$ of the pendulum becomes smaller and smaller as well.

It must be noted that the pendulum with energy $\mathcal{E}_n$ is not our original pendulum but a hypothetical pendulum to which we have iterated after n steps from the original. The iterated pendulums are linked by the special property that they have the same period per (9). In this sense we may regard the iterations as resulting in renormalized pendulums which are self-similar, thereby providing us with an example of the concept of renormalization, which is ubiquitous in quantum and statistical field theories.

Therefore, after a sufficient number of iterations we are able to consider the pendulum as effectively having a very small amount of energy, which in turn corresponds to small swings. But for small swings the pendulum acts as a simple harmonic oscillator for which the period can be found exactly. This period, as is well-known, is equivalent to 2π over the angular frequency corresponding to small librations. In this particular case, given the iterations, this angular frequency is none other than the $\lim_{n\to\infty}\omega_n = \lim_{n\to\infty}\Omega_n = AGM(\omega,\Omega)$. But due to the invariance of the integral in (9), the original period of the pendulum, the value of which we are interested in computing in the first place, remains the same during the iterations. Hence we conclude that the period of a pendulum described by ω and Ω can be found by,

$$\boxed{T = \frac{2\pi}{AGM(\omega,\Omega)}, \quad \omega = \sqrt{\frac{g}{l}}, \quad \Omega = \sqrt{\omega^2 - \frac{\mathcal{E}}{2}}, \quad 0 < \mathcal{E} < 2\omega^2.} \tag{10}$$

Thus, given the energy ($\mathcal{E}$) of the pendulum, its length (l), and the gravitational acceleration (g), we can find the period of the pendulum by first computing ω and Ω, and then iterating the two numbers via the arithmetic mean $\omega_{n+1} = AM(\omega_n,\Omega_n)$ and the geometric mean $\Omega_{n+1} = GM(\omega_n,\Omega_n)$ until the two sequences approach the common limit $AGM(\omega,\Omega)$. The iteration can be terminated when the desired degree of accuracy has been reached; the convergence to $AGM(\omega,\Omega)$ is very fast. Taking the reciprocal of that terminal value and multiplying it by 2π gives the period of the simple pendulum for any general libration. Table 1 shows several iterations for a pendulum with $\omega = 1\ s^{-1}$ and $\mathcal{E} = 1\ s^{-2}$, and hence $\Omega = \sqrt{1/2}\ s^{-1}$.

Iteration n	ω_n (s^{-1})	Ω_n (s^{-1})
0	1	$\sqrt{1/2}$
1	0.8535533905932737	0.7768869870150187
2	0.8152201888041462	0.7958228171106102
3	0.8055215029573781	0.8006574746586167
4	0.8030894888079974	0.8018725597212382
5	0.8024810242646178	0.8021767343016064
6	0.8023288792831120	0.8022528031856259

Table 1: Finding the period via iteration for a pendulum with $\omega = \omega_0 = 1\ s^{-1}$ and $\mathcal{E} = 1\ s^{-2}$, and therefore $\Omega = \Omega_0 = \sqrt{1/2}\ s^{-1}$. This scenario corresponds to a pendulum released from rest at a $90°$ angle to its downward vertical. (For a pendulum released from rest, $\mathcal{E} = \omega_0^2(1-\cos\theta_0)$ where θ_0 is the release angle.) The iterations yield $\omega_{n+1} = AM(\omega_n,\Omega_n)$ and $\Omega_{n+1} = GM(\omega_n,\Omega_n)$. At each successive iteration the values tend closer and closer to the arithmetic-geometric mean $AGM(\omega,\Omega)$, which to 3 decimal places can be taken as $0.802\ s^{-1}$. (These three first decimals are the same even after hundred iterations; in fact, ω_{27} and Ω_{27} agree to sixteen decimal places.) The period of the pendulum is thus $T \approx 2\pi/(0.802\ s^{-1}) = 7.8\ s$. If the small angle approximation is used, then $T = 2\pi/\omega_0 = 6.3\ s$; hence, the actual value for the period differs by about 24% relative to the small angle approximation.

Let us now turn to proving the invariance of the integrals in (9). We thus have to show that

$$\int_{-\infty}^{\infty} \frac{dy}{\sqrt{(\Omega^2+y^2)(\omega^2+y^2)}} = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{(GM(\omega,\Omega)^2+y^2)(AM(\omega,\Omega)^2+y^2)}}. \tag{11}$$

Let us first write the integral on the right hand side of (11) as,

$$\int_{-\infty}^{\infty} \frac{dz}{\sqrt{(\omega\Omega+z^2)\left[\left(\frac{\omega+\Omega}{2}\right)^2+z^2\right]}}.$$

Let us now make the following variable change:

$$z = \frac{1}{2}\left(y - \frac{\omega\Omega}{y}\right), \;\; 0 < y < \infty.$$

Thus $dz = \left(\frac{y^2+\omega\Omega}{2y^2}\right) dy$. By substitution,

$$\left(\frac{\omega+\Omega}{2}\right)^2 + z^2 = \frac{(\omega^2+y^2)(\Omega^2+y^2)}{4y^2},$$

$$\omega\Omega + z^2 = \frac{(\omega\Omega+y^2)^2}{4y^2}.$$

Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dz}{\sqrt{(\omega\Omega+z^2)\left[\left(\frac{\omega+\Omega}{2}\right)^2+z^2\right]}} &= \int_{0}^{\infty} \frac{(\omega\Omega+y^2)\cdot dy}{2y^2\left[\frac{(\omega^2+y^2)(\Omega^2+y^2)}{4y^2}\cdot\frac{(\omega\Omega+y^2)^2}{4y^2}\right]^{1/2}}, \\
&= \int_{0}^{\infty} \frac{2\cdot dy}{\sqrt{(\omega^2+y^2)(\Omega^2+y^2)}}, \\
&= \int_{-\infty}^{\infty} \frac{dy}{\sqrt{(\omega^2+y^2)(\Omega^2+y^2)}}.
\end{aligned}$$

This last expression is the left hand side of (11), thereby proving the invariance of the two integrals.

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