A Surprise in the Motion of the Simple Pendulum

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A simple harmonic oscillator such as a mass attached to a spring that oscillates back and forth horizontally without any dissipative forces such as friction is well-known to have the maximal speed (and minimal, that is, zero acceleration) as it passes the equilibrium point and the highest acceleration (and minimal, that is, zero speed) at the end points where the spring is maximally stretched. By analogy, then, a simple pendulum oscillating back and forth without any dissipative forces can similarly be expected to exhibit maximal speed (minimal acceleration) as it passes the lowest point of its circular arc and maximal acceleration (minimal speed) at the highest points of the arc where it reverses the direction of the swing. The goal of this essay is to investigate the speed and the acceleration of the simple pendulum closer than it is generally described in the literature. If the title of the essay is a give away, there is a counterintuitive surprise in store for us.

To see this, let us first establish the kinematical equations. To do so let us assume that the origin of the inertial frame $\mathcal{I}$ is at O where the pendulum is pivoted (see Figure 1). If we select a rotating polar frame $\mathcal{R}$ defined by its origin and the associated triad of unit vectors $\{O, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$ where $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_3$ (hence, for a right-handed system, $\mathbf{e}_3$ points perpendicularly out of the plane of the pendulum), then the position vector of the pendulum mass P is given by $\mathbf{r}_{P/O} = l\mathbf{e}_r$ where l is the fixed length of the pendulum.

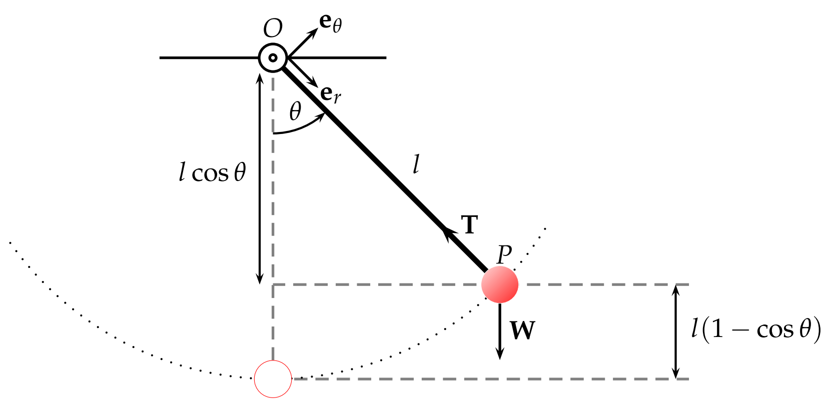

Figure 1: An instantaneous position (P) of the pendulum (of length l) with the unit vectors of the polar frame $\mathbf{e}_r$ and $\mathbf{e}_\theta$ marked on the oscillation plane. $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_3$ (hence, for a right-handed system, $\mathbf{e}_3$ points perpendicularly out of the plane of the pendulum). The coordinate frame has origin O, which is the pivot point. The potential energy of the pendulum is taken to be zero at the bottom of the swing. $\mathbf{T}$ and $\mathbf{W}$ are the tension on the weightless rod and the weight of the bob, respectively. | Drawing by AD.

The velocity of the pendulum can then be computed with respect to the inertial frame by taking the time derivative of $\mathbf{r}_{P/O}$ with respect to the inertial frame. Since $\mathbf{e}_r$ changes its direction with respect to the inertial frame as the pendulum swings, we obtain

$$^{\mathcal{I}}\mathbf{v}_{P/O} = \frac{^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} = l\frac{^{\mathcal{I}}d}{dt}\mathbf{e}_r = l\left(\frac{^{\mathcal{R}}d}{dt}\mathbf{e}_r + {^{\mathcal{I}}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{e}_r\right) = l\left(\mathbf{0} + \dot{\theta}\mathbf{e}_3 \times \mathbf{e}_r\right) = l\dot{\theta}\mathbf{e}_\theta, \tag{1}$$

where $^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$ is the angular velocity vector of the pendulum, which is equivalent to $\dot{\theta}\mathbf{e}_3$. The rate of change of the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_3$ with respect to the rotating frame $\mathcal{R}$ vanishes since these vectors are attached to it. Differentiating the above expression again with respect to time in the inertial frame, we obtain the acceleration vector as,

$$^{\mathcal{I}}\mathbf{a}_{P/O} = \frac{^{\mathcal{I}}d}{dt}\mathbf{v}_{P/O} = l\frac{^{\mathcal{I}}d}{dt}\left(\dot{\theta}\mathbf{e}_\theta\right) = -l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta, \tag{2}$$

where we have used

$$\frac{^{\mathcal{I}}d}{dt}\mathbf{e}_\theta = \left(\frac{^{\mathcal{R}}d}{dt}\mathbf{e}_\theta + {^{\mathcal{I}}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{e}_\theta\right) = \left(\mathbf{0} + \dot{\theta}\mathbf{e}_3 \times \mathbf{e}_\theta\right) = -\dot{\theta}\mathbf{e}_r.$$

Before proceeding, let us recapture the main results:

$$\boxed{\begin{aligned}&\mathbf{v}_{P/O} = l\dot{\theta}\mathbf{e}_\theta,\\ &\mathbf{a}_{P/O} = -l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta.\end{aligned}}$$

Now, considering the dynamics, the tension and weight vectors can be written as $\mathbf{T} = -T\mathbf{e}_r$ and $\mathbf{W} = mg\cos\theta\mathbf{e}_r - mg\sin\theta\mathbf{e}_\theta$, respectively. Combining the dynamics and kinematics via Newton's second law, where $\mathbf{F}_P = \mathbf{T} + \mathbf{W} = m{^{\mathcal{I}}}\mathbf{a}_{P/O}$, we obtain, after equating the respective coefficients of the unit vectors, the scalar equations of motion:

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta, \quad T = mg\cos\theta + ml\dot{\theta}^2.} \tag{3}$$

If the pendulum is released from rest at an initial angle θ_0 to the downward vertical where $0 < \theta_0 < \pi$, then at any other angle θ, by energy conservation, we have,

$$mgl(1-\cos\theta_0) = \frac{1}{2}mv^2 + mgl(1-\cos\theta),$$

from which we find,

$$v^2 = 2gl(\cos\theta - \cos\theta_0). \tag{4}$$

From (1) it follows that $v = l\dot{\theta}$. Substituting this expression in (4) yields,

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0).$$

Denoting the magnitude of the acceleration of the pendulum as a, it follows from (2) that

$$a^2 = l^2\dot{\theta}^4 + l^2\ddot{\theta}^2.$$

Substituting the expressions $\ddot{\theta} = -\frac{g}{l}\sin\theta$ and $\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0)$ in the expression for a^2 and simplifying, we arrive at,

$$a^2 = g^2\left(1 + 3\cos^2\theta - 8\cos\theta\cdot\cos\theta_0 + 4\cos^2\theta_0\right). \tag{5}$$

The speed and the magnitude of the acceleration of the pendulum can thus be summarized as,

$$\boxed{\begin{aligned} &v = \sqrt{2gl(\cos\theta - \cos\theta_0)}, \quad 0 < \theta_0 < \pi & (6)\\ &a = g\sqrt{(1 + 3\cos^2\theta - 8\cos\theta\cdot\cos\theta_0 + 4\cos^2\theta_0)}. & (7)\end{aligned}}$$

(Note that only the positive square roots are considered given our interest in the magnitudes.) An important observation immediately stands out: which is, that the speed of the pendulum varies as the square root of the length of the pendulum while the magnitude of the acceleration is independent of the length of the pendulum. At $\theta = \pm\theta_0$ the speed clearly vanishes. That is, at the end points of motion on either side of the downward vertical the pendulum comes to a stop before reversing its course. For the speed to attain its highest value, $\cos\theta$ has to be a maximum. Therefore, the speed is highest when $\cos\theta = 1$ which in turn implies that $\theta = 0$. Thus the pendulum attains the maximum speed at the bottom of its swing. This latter conclusion also follows by differentiating v^2 with respect to θ and setting it to zero, which yields a stationary point at $\theta = 0$. Taking the second derivative of v^2 with respect to θ it follows that the speed at $\theta = 0$ is a maximum since,

$$\left.\frac{d^2}{d\theta^2}v^2\right|_{\theta=0} = -2gl < 0.$$

The reader may consider these observations as intuitive. This intuition regarding pendulum's motion may breakdown as we consider its acceleration. To see this, let us compute where the minima or the maxima of acceleration occur during pendulum's motion. Differentiating (5) with respect to θ we obtain,

$$\frac{d}{d\theta}a^2 = 2g^2\sin\theta(4\cos\theta_0 - 3\cos\theta).$$

Thus, stationary points in acceleration occur at either $\sin\theta = 0$ or at $\cos\theta = (4/3)\cos\theta_0$. These points correspond to $\theta = 0$ and $\theta = \cos^{-1}[(4/3)\cos\theta_0]$, respectively. Taking the second derivative of a^2 with respect to θ, we obtain,

$$\frac{d^2}{d\theta^2}a^2 = 2g^2\left[4\cos\theta_0\cos\theta + 3(\sin^2\theta - \cos^2\theta)\right].$$

Therefore,

$$\left.\frac{d^2}{d\theta^2}a^2\right|_{\theta=0} = 2g^2\left[4\cos\theta_0 - 3\right].$$

For a to be a minimum at $\theta = 0$ the left side of the above expression has to be > 0. This in turn implies that $\cos\theta_0 > 3/4$, or in other words, $\theta_0 < \cos^{-1}(3/4)$. The value of $\cos^{-1}(3/4) \approx 41.41^\circ \approx 0.72$ rad. Thus the acceleration of the pendulum is a minimum at the bottom of the swing if the pendulum is released at an angle (to the downward vertical) below $\cos^{-1}(3/4) \approx 41.41^\circ \approx 0.72$ rad. Consequently, the acceleration a is a maximum at $\theta = 0$ when $\theta_0 > \cos^{-1}(3/4)$. The behavior of acceleration is therefore not as intuitive as that of speed of the pendulum and whether it is a minimum or a maximum at the bottom of the swing depends on its release angle. In summary, if the pendulum is released from rest below the critical angle $\theta_c = \cos^{-1}(3/4) \approx 41.41^\circ$ then its acceleration is a minimum at the bottom of the swing; if it is released above the critical angle θ_c, then the acceleration attains a maximum at the bottom of the swing.

At the stationary point $\theta = \cos^{-1}[(4/3)\cos\theta_0]$,

$$\left.\frac{d^2}{d\theta^2}a^2\right|_{\theta=\cos^{-1}[(4/3)\cos\theta_0]} = 2g^2\left[3 - (16/3)\cos^2\theta_0\right].$$

For a to be a minimum at $\theta = \cos^{-1}[(4/3)\cos\theta_0]$ the left side of the above expression has to be > 0. This in turn implies that $\cos\theta_0 < 3/4$, or in other words, $\theta_0 > \cos^{-1}(3/4)$. For $\cos\theta = (4/3)\cos\theta_0$ to hold we must have $(4/3)\cos\theta_0 < 1$ (we rule out $(4/3)\cos\theta_0 = 1$ since we take $\theta > 0$). This condition again leads to $\cos\theta_0 < 3/4$. Thus the magnitude of the acceleration is a minimum at $\theta = \cos^{-1}[(4/3)\cos\theta_0]$ when $\theta_0 > \cos^{-1}(3/4)$; that is when the release angle is above the critical angle θ_c. At the stationary point $\theta = \cos^{-1}[(4/3)\cos\theta_0]$ there is no solution for a that maximizes it. Also, at $\theta = 0$ or at $\theta = \cos^{-1}[(4/3)\cos\theta_0]$, a is neither a minimum nor a maximum if $\theta_0 = \theta_c = \cos^{-1}(3/4)$. At $\theta = \theta_0$ the magnitude of acceleration is simply $a = g\sin\theta_0$. The reason for this is clear: at $\theta = \theta_0$ the angular speed $\dot{\theta} = 0$ and therefore only the tangential ($\mathbf{e}_\theta$) component of the acceleration remains, which is equivalent to $g\sin\theta_0$ in magnitude. We summarize these results in the next page.

Summary of the dynamics

1. $v = 0$ at $\theta = \theta_0$ for all release angles θ_0.
2. $v_{\text{max}} = \sqrt{2gl(1 - \cos\theta_0)}$ at $\theta = 0$ for all release angles θ_0.
3. $a = g\sin\theta_0$ at $\theta = \theta_0$ for all release angles θ_0.
4. $a_{\text{min}} = 2g(1 - \cos\theta_0)$ at $\theta = 0$ if $\theta_0 < \theta_c = \cos^{-1}(3/4)$.
5. $a_{\text{max}} = 2g(1 - \cos\theta_0)$ at $\theta = 0$ if $\theta_0 > \theta_c = \cos^{-1}(3/4)$.
6. $a_{\text{min}} = g\sqrt{1 - \frac{4}{3}\cos^2\theta_0}$ at $\theta = \cos^{-1}[(4/3)\cos\theta_0]$ if $\theta_0 > \theta_c = \cos^{-1}(3/4)$.
7. a is neither a minimum nor a maximum at $\theta = 0$ if $\theta_0 = \theta_c = \cos^{-1}(3/4)$.

Figures 2 and 3 show the variation of speed and the magnitude of the acceleration of the simple pendulum against θ for several release angles θ_0.

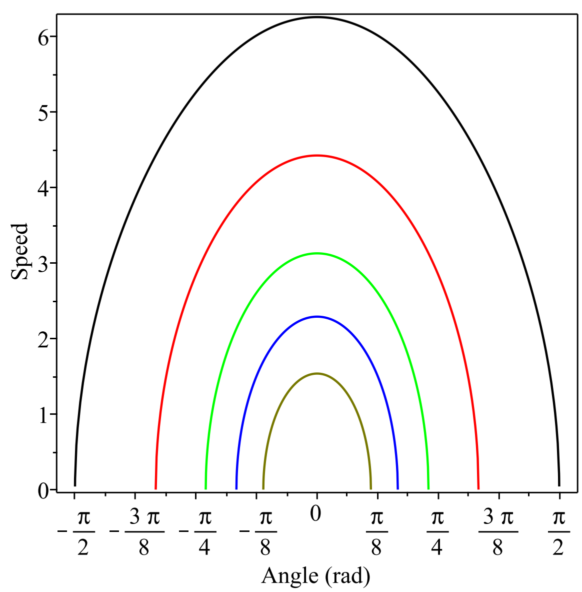

Figure 2: The variation of the pendulum speed v (in ms^{-1}) against angle θ (measured with respect to the downward vertical in radians) for several release angles θ_0 with $l = 2$ m and $g = 9.81$ ms^{-2} when the pendulum has been released from rest: black ($\theta_0 = \pi/2$ rad $= 90°$), red ($\theta_0 = \pi/3$ rad $= 60°$), green ($\theta_0 = \theta_c = \cos^{-1}(3/4) \approx 41.41°$), blue ($\theta_0 = \pi/6$ rad $= 30°$), olive ($\theta_0 = \pi/9$ rad $= 20°$). The speed is a maximum at $\theta = 0$ – that is, at the bottom of the swing – regardless of the release angle θ_0.

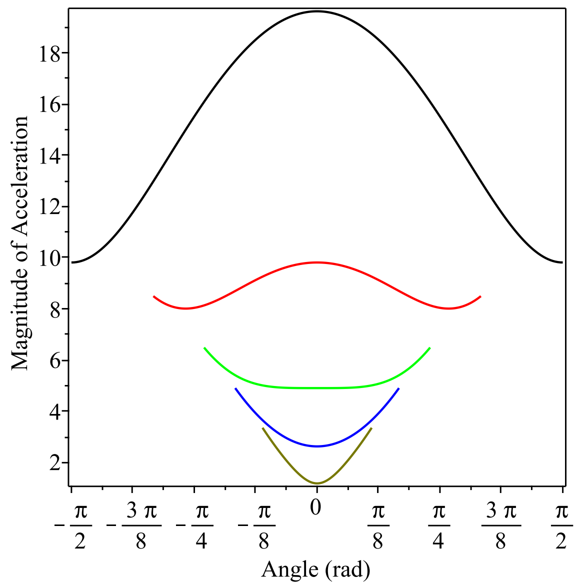

Figure 3: The variation of the pendulum's magnitude of acceleration a (in ms^{-2}) against angle θ (measured with respect to the downward vertical in radians) for several release angles θ_0 with $g = 9.81$ ms^{-2} when the pendulum has been released from rest: black ($\theta_0 = \pi/2$ rad $= 90°$), red ($\theta_0 = \pi/3$ rad $= 60°$), green ($\theta_0 = \theta_c = \cos^{-1}(3/4) \approx 41.41°$), blue ($\theta_0 = \pi/6$ rad $= 30°$), olive ($\theta_0 = \pi/9$ rad $= 20°$). a is a minimum at $\theta = 0$ – that is, at the bottom of the swing – when $\theta_0 < \theta_c = \cos^{-1}(3/4)$ (olive and blue curves). This minimum turns into a maximum at $\theta = 0$ when $\theta_0 > \theta_c = \cos^{-1}(3/4)$ (red and black curves). In this latter case, a is minimized at $\theta = \pm\cos^{-1}[(4/3)\cos\theta_0]$ (red and black curves). a is neither a minimum nor a maximum at $\theta = 0$ if $\theta_0 = \theta_c = \cos^{-1}(3/4)$ (green curve).

Further Reading: For establishing the equations of motion as done in this essay, the book *Engineering Dynamics: A Comprehensive Introduction* (Princeton University Press, 2011) by Jeremy Kasdin and Derek Paley is highly recommended. It follows a very systematic approach in setting up reference frames, coordinate systems, and finding equations of motions of physical systems with great clarity.

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