The Driven Simple Pendulum

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1 Introduction

This essay concerns the driven simple pendulum, which consists of a pendulum that is free to librate or rotate about the pivot point on the vertical plane but where the pivot is being driven along the horizontal axis through it by some mechanism at the same time. It is important to clarify, that by a "driven pendulum", we mean a pendulum where its pivot point is driven, which then secondarily imparts the effects of the driving mechanism to the motion of the pendulum mass itself. We first set out to obtain the general equations of motion and then specialize to the scenario where the driving mechanism of the pivot is defined by simple harmonic oscillations. In the course of this we will encounter a system exhibiting chaotic motion.

Consider Figure 1 which depicts the general set up of the system.

The inertial frame against which we track motion has the origin at O. The pivot point P is constrained to move along the X-axis, and at the particular instance shown it is at a distance x from O. Thus x can vary independently with time according to the specific driving mechanism imparted to it. The pendulum of length l and mass m is attached to the pivot point P and is free to librate or rotate about P on the vertical plane. The angle θ to the downward vertical can also vary independently with time. Thus the driven simple pendulum is a system with two degrees of freedom having the generalized coordinates (x, θ). At any given instance the coordinate (X, Y) of the pendulum mass on the vertical plane is thus,

$$X = x + l \sin\theta, \; Y = -l\cos\theta.$$

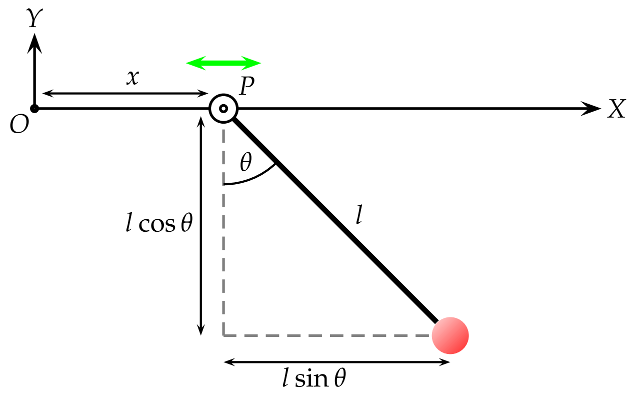


Figure 1: Coordinate set up for establishing the Lagrangian for the driven simple pendulum where the pivot is constrained to move along the X-axis. | Drawing by AD.

Thus,

$$\dot{X} = \dot{x} + l\dot{\theta}\cos\theta, \; \dot{Y} = l\dot{\theta}\sin\theta.$$

The kinetic energy ($\mathcal{E}_k$) and the potential energy ($\mathcal{E}_p$) are then given by,

$$\mathcal{E}_k = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2), \; \mathcal{E}_p = mgY,$$

where g is the gravitational acceleration constant.

Substituting the expressions, the Lagrangian ($\mathcal{L} = \mathcal{E}_k - \mathcal{E}_p$) takes the form,

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + \dot{\theta}^2 l^2\right) + mgl\cos\theta.$$

The equation of motion with respect to θ is then given by the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \left(\frac{\partial\mathcal{L}}{\partial\theta}\right) = 0,$$

which yields,

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{\ddot{x}}{l}\cos\theta.} \tag{1}$$

2 Pendulum Driven at Constant Velocity

If the pivot point is moving at constant velocity along the X-axis, then $\ddot{x} = 0$, which results in the classic expression for the simple pendulum: $\ddot{\theta} = -(g/l)\sin\theta$. The fact that the equation of motion that results when the pendulum's pivot is stationary also holds when the pivot is moving at constant velocity is simply a consequence of the Galilean principle of relativity, which states that the laws of physics are the same for inertial observers. That is, when the pivot is moving at a constant velocity (less than the speed of light, of course!), we can find an inertial frame (equivalently, an inertial observer) in which the pivot is at rest. Therefore, according to the Galilean principle of relativity, there is no distinction in the observed dynamics of the pendulum between an inertial observer for whom the pivot is moving at a constant velocity and an inertial observer who is moving at the same velocity as that of the pivot relative to the stationary (inertial) observer. Thus the same equation of motion for the pendulum's motion must result regardless of whether the pivot is at rest with respect to an inertial observer or moving at constant velocity with respect to that observer. Another way to state this is, the fact that (1) does not depend on $\dot{x}$ is a consequence of the Galilean principle of relativity. If (1) depended on $\dot{x}$, then inertial observers moving at different constant velocities will observe different dynamics of the pendulum, and hence they can use the pendulum to distinguish their motion from all other inertial observers; in that case, the notion that all inertial frames are equivalent with respect to laws of physics will disappear.

3 Pendulum Driven at Constant Acceleration

Let us now consider the case where the pivot is moving along the positive X direction at a constant acceleration $\mathbf{a}$: that is $\ddot{x} = +a$, a constant. Let us also assume that this acceleration was achieved starting from rest. Then, at the start, when the pivot is at rest, the pendulum is also at rest in stable equilibrium hanging down vertically. As the constant acceleration a is achieved by the pivot, the pendulum moves in the opposite direction to that of the movement of the pivot and settles at an angle that corresponds to the new stable equilibrium. Since the pivot is moving in the positive X direction in our case, this means that the pendulum would swing clockwise and settle at a new angle. The reason the pendulum would move in the opposite direction to the (horizontal) acceleration of the pivot can be conceptually grasped as follows: since the pendulum mass is connected to the pivot by the rod, when the pivot accelerates horizontally in a particular direction, the pendulum mass will also have the same acceleration in the same direction as that of the pivot.

Now if we consider the forces acting on the pendulum mass, we know that the gravitational force acts vertically downward, and therefore, its contribution in any horizontal direction is zero. But some other force must then supply the necessary horizontal acceleration to the pendulum mass. There is only one other force that can provide this acceleration, and that is the tension force on the pendulum that acts along the rod. If this force is acting only vertically upwards to counterbalance the force of gravity on the pendulum mass, then its contribution in any horizontal direction is also zero. Therefore, the only way that this tension force can impart a horizontal acceleration in the same direction as that of the pivot is for the pendulum to swing in the opposite direction to that of the movement of the pivot, thereby providing the necessary horizontal force component to the pendulum mass. An observer that is situated in the same reference frame as the accelerated pivot will also conclude that the pendulum has swung in the opposite direction to that of the movement of the pivot thus providing an absolute way of distinguishing whether one is in an accelerated frame of reference or not. The situation when the pendulum has settled into the new stable equilibrium when the pivot is accelerating to the right is shown in Figure 2.

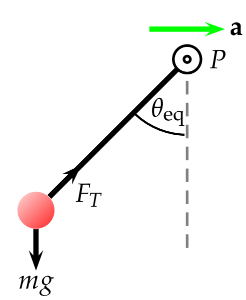


Figure 2: When the pivot moves at constant acceleration **a** to the right with respect to an inertial frame, the pendulum settles at an angle $-\theta_{\text{eq}}$; that is, to the left of the downward vertical. The horizontal force component of the tension force (F_T) on the pendulum provides the necessary force on the pendulum mass to achieve the rightward acceleration **a**. | Drawing by AD.

We now set out to find the angle to the downward vertical that the pendulum settles when the pivot is moving at constant acceleration. Let us assume that the pendulum is in stable equilibrium at angle $+\theta_{\text{eq}}$ to the downward vertical when the pivot is moving horizontally in the positive X direction at constant acceleration a (see Figure 3).

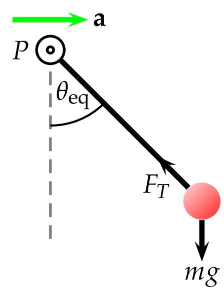


Figure 3: Diagram for deriving the expression (2) for the equilibrium angle (θ_{eq}) when the pivot of the pendulum moves at a constant acceleration where the sign convention is: $\overrightarrow{a} : (+) \mid \overleftarrow{a} : (-)$, so that the equilibrium angle is positive if the pendulum is to the right of the downward vertical, and negative if it is to the left of the downward vertical. The tension on the pendulum mass is F_T. | Drawing by AD.

The angle is positive if the pendulum is to the right of the downward vertical, and negative if it is to the left of the downward vertical (in other words, we take angles measured counterclockwise to the downward vertical as positive). Applying Newton's second law in the positive X and positive Y directions, respectively, we obtain,

$$\rightarrow -F_T \sin\theta_{\text{eq}} = ma, \uparrow F_T \cos\theta_{\text{eq}} = mg.$$

Dividing the first expression by the second and solving for θ_{eq}, we arrive at,

$$\boxed{\theta_{\text{eq}} = \tan^{-1}\left(-\frac{a}{g}\right), \quad \overrightarrow{a} : (+) \mid \overleftarrow{a} : (-).} \tag{2}$$

The new stable equilibrium for the pendulum thus depends on the acceleration of the pivot and the gravitational acceleration. Given that we have taken a to be in the positive X direction [i.e., to the right; denoted in (2) by $\overrightarrow{a} : (+)$], a negative θ_{eq} would imply that the pendulum has settled to the left of the downward vertical. Similarly, if a is negative, that is, if the pivot is moving in the negative X direction [i.e., to the left; denoted in (2) by $\overleftarrow{a} : (-)$], then a positive θ_{eq} would result, implying that the pendulum has settled to the right of the downward vertical.

If we release the pendulum from rest at a new angle θ_0 when the pivot has constant acceleration, then it will librate around the line through the pivot that is at an angle θ_{eq} to the downward vertical; on which side of the downward vertical the line would be is determined by the direction of the acceleration of the pivot point: the line would be to the left of the downward vertical if the pivot is accelerating to the right and vice versa. The resulting phase curves for a specific set of parameters are shown in Figure 4. As can be seen from the figure, the closed phase curves are centered at $\theta_{\text{eq}} = \tan^{-1}(-a/g)$. The outermost phase curve for librations, the separatrix, is bounded between $\theta = \theta_{\text{eq}} \pm \pi$, which corresponds to the same point in physical space, which is an unstable equilibrium point for the pendulum.

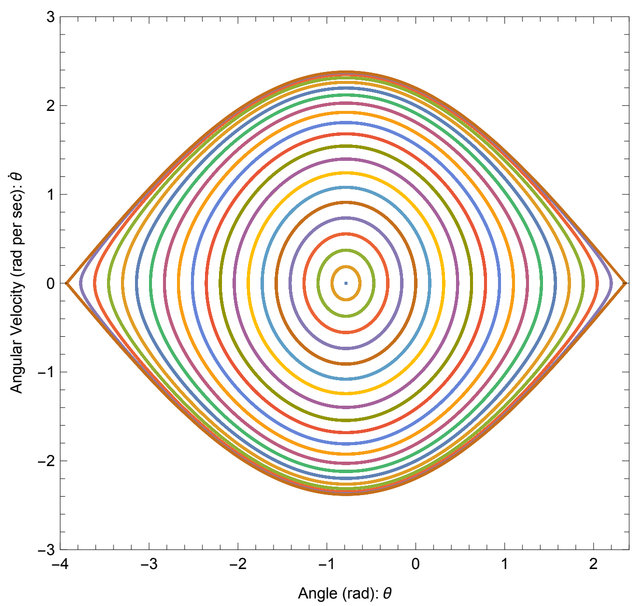


Figure 4: Phase portrait of a simple pendulum whose pivot point has constant acceleration $a = +9.8\ \text{ms}^{-2}$ (i.e., to the right in the positive X direction); $l = 9.8$ m and $g = 9.8\ \text{ms}^{-2}$. The pendulum now has a stable equilibrium at $\theta_{\text{eq}} = \tan^{-1}(-a/g) = \tan^{-1}(-1) = -\pi/4$ rad ≈ -0.785 rad. The center of the phase portrait thus occurs at $-\pi/4$ rad. In other words, the stable equilibrium for the pendulum occurs at an angle $\pi/4$ rad to the left of the downward vertical since its pivot has constant acceleration to the right. If the pendulum is released from rest at a different angle to that of θ_{eq}, then it librates around the line through the pivot that is at an angle θ_{eq} to the downward vertical. The closed phase curves centered at θ_{eq} demonstrate this libration for several release positions. The outermost phase curve for librations, the separatrix, is bounded between $\theta = \theta_{\text{eq}} \pm \pi = -\pi/4 \pm \pi$, which corresponds to the same point in physical space. Hence the pendulum is in unstable equilibrium at $\theta = -\pi/4 \pm \pi$.

4 Pendulum Driven in Simple Harmonic Oscillations: Ordered Motion

We have so far considered the situations where $\ddot{x} = 0$ or $\ddot{x} = a =$ constant in (1); that is, when the pivot point is moving at constant velocity or when it is moving at constant acceleration. The dynamics of the pendulum under such conditions is not fundamentally different from when the pivot is at rest relative to an inertial frame. However, fundamentally different dynamics arise when the pivot point is driven according to, say, simple harmonic oscillations: in that case $\ddot{x} = -\Omega^2 x$, where Ω is the angular frequency of the oscillations of the pivot point. Note that the pivot point is still constrained to move only along the X-axis.

Let us propose the solution $x(t) = A\cos\Omega t$ to the equation $\ddot{x} = -\Omega^2 x$. Thus, at $t = 0$, $x(0) = A$; therefore, the amplitude of the simple harmonic oscillations of the pivot point is A. The velocity of the pivot point is: $\dot{x}(t) = -A\Omega\sin\Omega t$; therefore, at $t = 0$, $\dot{x}(0) = 0$. So the pivot point can be considered as released from rest at $x = A$. As such, the pendulum mass now librates or rotates about the pivot point on the vertical plane while the pivot point oscillates simple harmonically between $x = \pm A$ centered about $x = 0$. We therefore have $\ddot{x} = -A\Omega^2\cos\Omega t$. Substituting this expression in (1), the equation of motion for the simple harmonically driven pendulum becomes,

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{A}{l}\Omega^2\cos\theta\cdot\cos\Omega t.} \tag{3}$$

We therefore see that the equation of motion is explicitly time dependent, which gives rise to fundamentally different dynamics in contrast to the previous two scenarios. We will have to utilize numerical integration to solve (3) to obtain the angle (θ) and the angular velocity ($\dot{\theta}$) as functions of time t.

When the pivot is driven simple harmonically, the pendulum may exhibit regularity or chaotic motion depending on the initial conditions and the driving angular frequency Ω. Figure 5 shows the phase space of the pendulum with the initial conditions $\theta_0 = \pi/4$ rad and $\dot{\theta}_0 = -1$ rad s^{-1} along with the parameters $\Omega = 0.5$ rad s^{-1}, $A = l = 9.8$ m, $g = 9.8$ ms^{-2}. This phase space was created by determining the angle and the angular velocity of the pendulum at positive integer intervals of the period of the simple harmonic oscillations of the pivot point. The collection of these snapshots of the phase space is called a *Poincaré Section*. As we see from Figure 5 the Poincaré section of the harmonically driven simple pendulum shows regularity for this particular set of initial conditions. In other words, if $T = 2\pi/\Omega$, and at $t = 3T$: $\theta(3T) = \theta_3$ and $\dot{\theta}(3T) = |\dot{\theta}_3|$, then if at $t = 7T$ $\theta(7T) = \theta_3$, then $\dot{\theta}(7T) = |\dot{\theta}_3|$. Figure 6 shows the phase portrait as in Figure 5 but with the time dimension also added in.

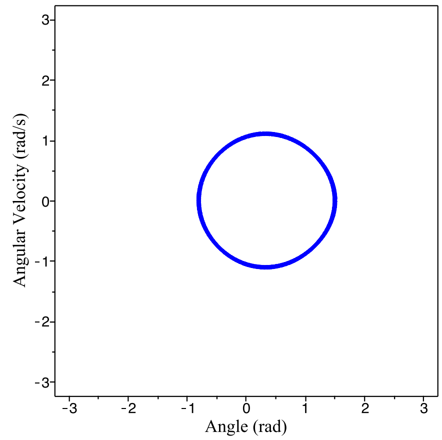


Figure 5: Phase portrait of the simple pendulum whose pivot point is driven according to $\ddot{x} = -\Omega^2 x$. The initial conditions for this particular case is such that $\theta_0 = \pi/4$ rad and $\dot{\theta}_0 = -1$ rad s^{-1} with the parameters $\Omega = 0.5$ rad s^{-1}, $A = l = 9.8$ m, $g = 9.8$ ms^{-2}. This phase space was created by determining the angle and the angular velocity of the pendulum at positive integer intervals of the period ($T = 2\pi/\Omega$) of the simple harmonic oscillations of the pivot point. The collection of these snapshots of the phase space is called a Poincaré section and shows regularity for the specific set of initial conditions and parameters chosen.

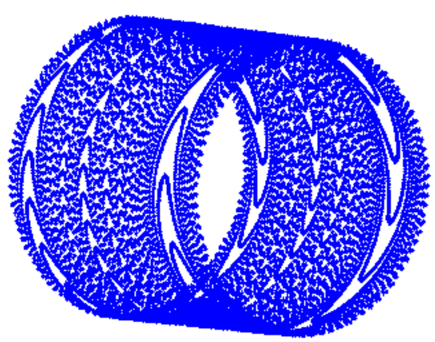


Figure 6: Phase portrait of the simple pendulum whose pivot point is driven according to $\ddot{x} = -\Omega^2 x$. The phase portrait is the same as in that shown in Figure 5 but with the time dimension also added in. The two-dimensional circular cross section in the (angle, angular velocity) phase space then becomes the three-dimensional cylinder in the (angle, angular velocity, time) space.

5 Pendulum Driven in Simple Harmonic Oscillations: Chaotic Motion

We now describe a set of initial conditions and parameters under which the pendulum exhibits disorderly or chaotic motion when the pivot is driven simple harmonically. This specific set of initial conditions and the parameters are: $\theta_0 = \pi$ rad and $\dot{\theta}_0 = -1$ rad s^{-1}, $\Omega = 1$ rad s^{-1}, $A = l = 9.8$ m, $g = 9.8$ ms^{-2}. Thus, compared to the previous case, the only differences are that we release the pendulum (at non-zero angular velocity) hanging upside down at $\theta_0 = \pi$ rad (instead of at $\theta_0 = \pi/4$ rad) and we drive the pendulum harmonically at $\Omega = 1$ rad s^{-1} (instead of at $\Omega = 0.5$ rad s^{-1}). The resulting phase portrait is shown in Figure 7.

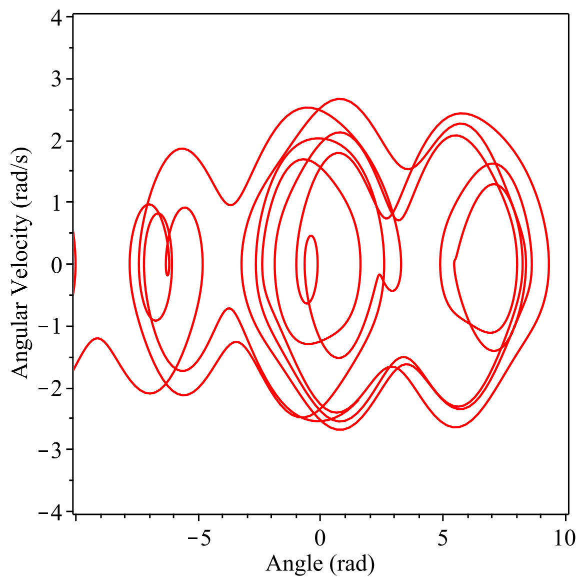


Figure 7: Phase portrait of the simple pendulum whose pivot point is driven according to $\ddot{x} = -\Omega^2 x$. The initial conditions for this particular case is such that $\theta_0 = \pi$ rad and $\dot{\theta}_0 = -1$ rad s^{-1} with the parameters $\Omega = 1$ rad s^{-1}, $A = l = 9.8$ m, $g = 9.8$ ms^{-2}. Unlike the orderly phase portraits we have seen when the pendulum's pivot is stationary, or moving at constant velocity or acceleration, or even under harmonic oscillations with the set of initial conditions and parameters discussed in the previous sections, it is clear that this specific initial conditions and parameters give rise to chaotic dynamics. Note that this phase portrait is *not* a Poincaré section.

Given the disorderly nature of the orbit of the pendulum in phase space, we can expect its Poincaré section to not show the regularity as was seen in Figure 5. This is indeed what we see in Figure 8. The pendulum may return to the same angular position with respect to the downward vertical but its angular velocity is not guaranteed to have the same magnitude as it had before in an earlier cycle. In essence, the phase points show a sense of randomness on the Poincaré section.

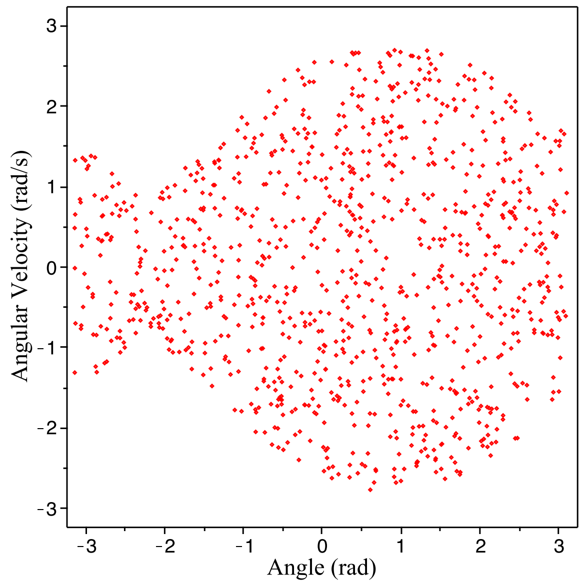
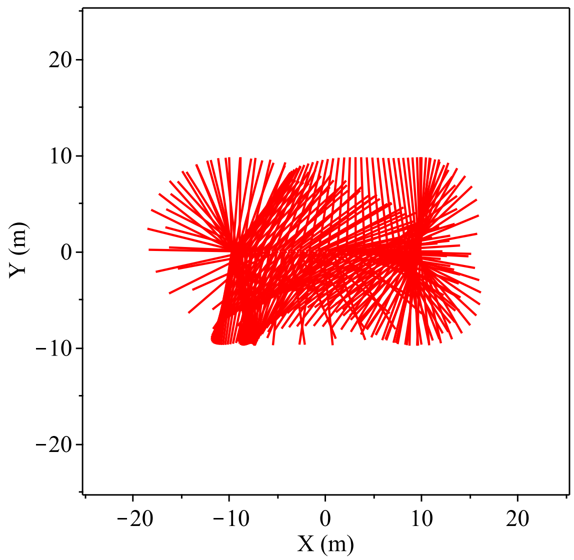


Figure 8: The Poincaré section for the simple pendulum whose phase portrait is depicted in Figure 7. Unlike the regularity seen in the Poincaré section in Figure 5, we see that the pendulum may return to the same angular position with respect to the downward vertical but its angular velocity is not guaranteed to have the same magnitude as it had before in an earlier cycle. The phase points show a sense of randomness on the Poincaré section.

When the dynamics of a system enters the chaotic regime its motion depends sensitively on the initial conditions. We therefore can expect to have the harmonically driven simple pendulum released from nearly identical initial conditions yet observe widely differing subsequent motions. Figure 9 shows snapshots of the pendulum rod when released from nearly identical initial conditions. For the nearly identical initial conditions described in the text accompanying Figure 9, Figure 10 shows the orbits of the pendulum masses.

The seemingly random set of points in the Poincaré section shown in Figure 8 starts to show structure when a damping effect is added to the equation of motion. If we assume the contribution of damping to the acceleration of the pendulum to be of form $-\mu\dot{\theta}$, then the equation of motion (3) becomes

$$\boxed{\ddot{\theta} = -\mu\dot{\theta} - \frac{g}{l}\sin\theta + \frac{A}{l}\Omega^2\cos\theta\cdot\cos(\Omega t + \phi),} \tag{4}$$



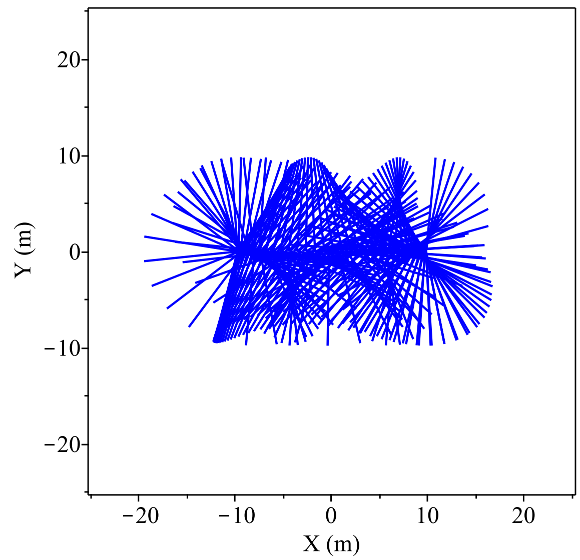


Figure 9: Snapshots of the pendulum rod in physical space for two nearly identical initial conditions: $\theta_0 = \pi$ rad and $\dot{\theta}_0 = -1$ rad s^{-1} (left, red); $\theta_0 = \pi - 0.1$ rad and $\dot{\theta}_0 = -1$ rad s^{-1} (right, blue). The parameters $\Omega = 1$ rad s^{-1}, $A = l = 9.8$ m, $g = 9.8$ ms^{-2} are common to both cases. Each second of the motion contains 10 snapshots. Though the initial conditions are nearly identical – only the starting angle differs between the two scenarios by about 3.2% – the subsequent motions diverge significantly from each other. Therefore, when the pendulum is driven simple harmonically, it may exhibit sensitive dependence on initial conditions, which is a characteristic of chaotic motion.

where we have also included the phase factor ϕ. The resulting Poincaré sections for a series of phases is shown in Figure 11. Not surprisingly, the Poincaré sections obtained depend on the initial phase of the system.

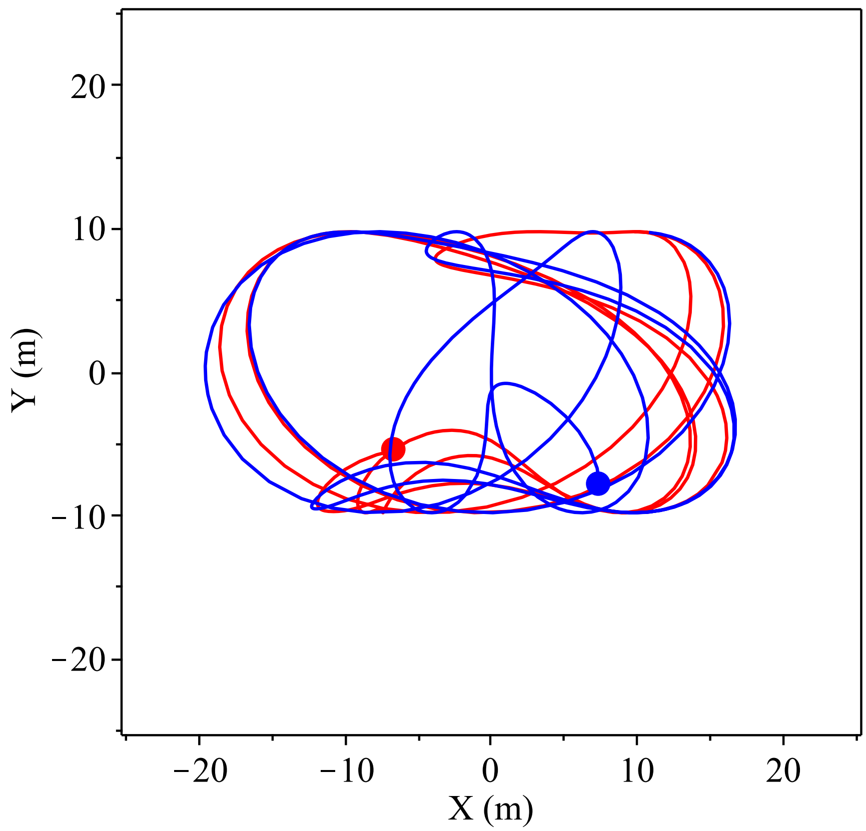


Figure 10: The orbits of the pendulum masses in physical space tracked for 30 s when released simultaneously from nearly identical initial conditions as described in the text accompanying Figure 9. Though the initial conditions are nearly identical – only the starting angle differs between the two scenarios by about 3.2% – the subsequent motions diverge significantly from each other after about 4 s. Therefore, when the pendulum is driven simple harmonically, it may exhibit sensitive dependence on initial conditions, which is a characteristic of chaotic motion.

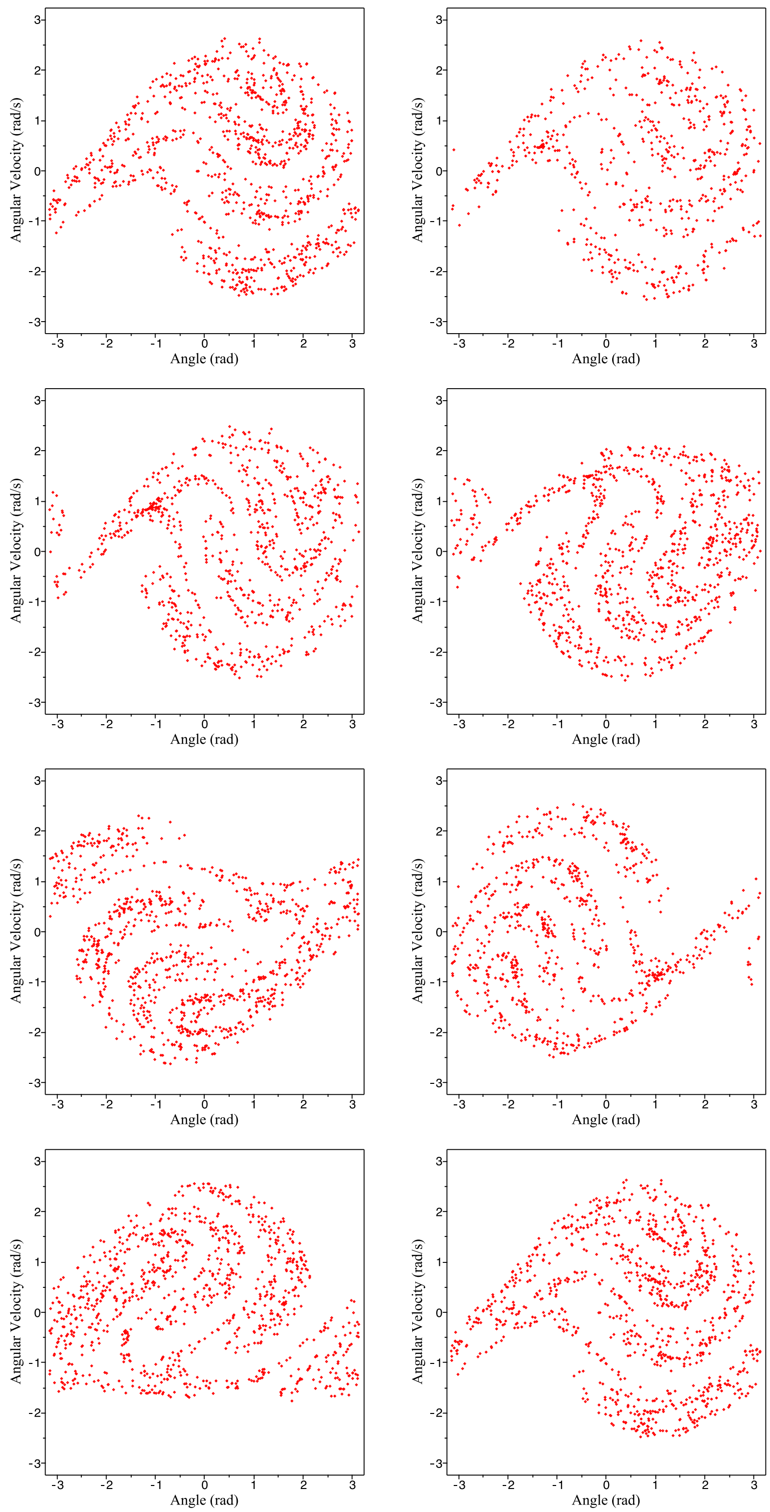


Figure 11: The Poincaré sections of the driven pendulum when damping is added to the motion. Each section depicts the resulting motion when started from a specific phase ϕ of the driving motion with the sequence from top left to bottom right corresponding to $\phi = \frac{n}{10}\pi$ where $n = 0, 1, 2, 4, 8, 12, 16, 20$. The first and the last sections differ by a phase of 2π, therefore, they are identical. The parameters are set at $\Omega = 1$ rad s^{-1}, $A = l = 9.8$ m, $g = 9.8$ ms^{-2}, $\mu = 0.05$ s^{-1}. The initial conditions were set at $\theta_0 = \pi$ rad and $\dot{\theta}_0 = -1$ rad s^{-1}.

♣