From Pendulum Swings to Universal Gravitation: A Theorem of Bohlin

Rasil Warnakulasooriya

In establishing the equation of motion $\ddot{\theta} = -(g/l)\sin\theta$ for a pendulum that swings on a vertical plane (where θ is the angle measured from the downward vertical, l is the length of the pendulum rod, and g is the constant gravitational acceleration), the inverse square (i.e., $1/r^2$) nature of the gravitational force does not enter the picture. In this essay we embark to demonstrate the beautiful connection that exists between small swings of the pendulum and the universal law of gravitation. The connection that we speak of is the following: Imagine a particle moving under the attractive force $F \propto -r$. Then, this force is related to the inverse square gravitational attraction given by $\mathcal{F} \propto -\frac{1}{r^2}$. In other words, it can be shown that F and $\mathcal{F}$ are dual forces in the sense that if a particle is moving in an orbit under the force field $F \propto -r$, then the only other force field that will make the particle traverse an orbit that is similar to the former has the form $\mathcal{F} \propto -\frac{1}{r^2}$. It turns out that for the particular F and $\mathcal{F}$ the orbits are elliptical, which are familiar from the orbits of the planets about the Sun. Another way to cast this duality is to say that F and $\mathcal{F}$ are the only two forms of the forces that give rise to elliptical orbits!

To demonstrate the duality of the force fields F and $\mathcal{F}$ we first make the connection between the small swings of the pendulum given by the equation of motion $\ddot{\theta} = -\theta$ (where the factor g/l is taken as 1 for simplicity without loss of generality) with that of $F \propto -r$, where r is the distance measured from the center of the force field F. We then operate in the complex plane where the demonstration of the connection between F and $\mathcal{F} \propto -\frac{1}{r^2}$ can be made in a more direct way. This is in contrast to the geometrical approach one can take in the real plane, which, while illuminating, would take us somewhat off-course in our main narrative. However, we will touch upon the geometrical idea behind this connection due to Newton. The duality of the above two forces is encapsulated in a theorem by Bohlin.[†]

Toward establishing the connection between small swings of the pendulum with that of $F \propto -r$, consider Figure 1. If we imagine the shadow that would be cast by the pendulum bob on a horizontal plane under it, then, as the pendulum swings, its shadow will move back and forth on a straight line on the horizontal plane.

[†]I first came across this beautiful result during a reading of V. I. Arnol'd's delightful book *Huygens & Barrow, Newton & Hooke* (Birkhäuser, Basel, 1990; translated to English from the Russian by E. Primrose). This essay is a detailed elaboration of the discussion given there (a more general result for dual forces developed by Kasner and by Arnol'd can be found in the book as well). Although I have not read, the reference to Bohlin's work as cited by Arnol'd is: K. Bohlin, *Bulletin Astronomique*, **28** (1911), 144. Readers are encouraged to read Arnol'd's book, which contains many little gems on physics, mathematics, history, and personalities, as usually is the style of Arnol'd's more popular writing.

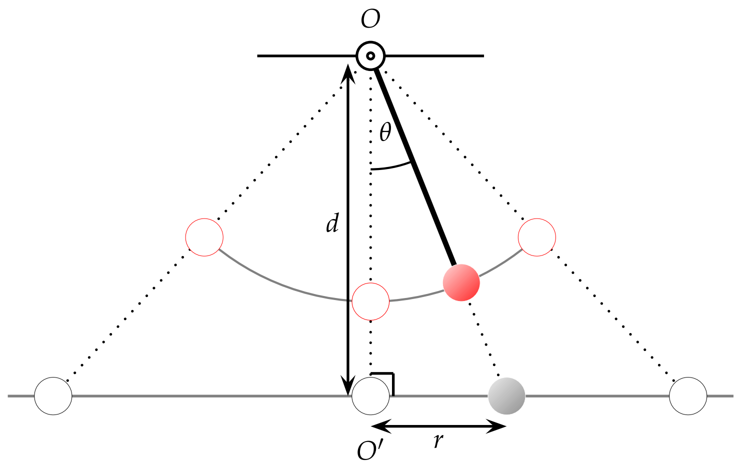


Figure 1: As the pendulum bob (red) makes small swings about the pivot O, its shadow (gray) makes back and forth simple harmonic movements about O' along the horizontal line. Thus the equation of motion of the pendulum for small swings, $\ddot{\theta} = -\theta$, leads to the equation of motion of its shadow $\ddot{r} = -r$. (Here the constant factor g/l is taken as 1 without loss of generality.) | Drawing by AD.

The movement of the shadow will appear as if there is an attractive center of force at O' and that the attraction on the shadow mass grows the further it is from the center. (And, just as the actual pendulum comes to a stop before reversing its direction of swing, the shadow mass also comes to a stop on the line before reversing its direction.) Thus, the shadow mass executes simple harmonic motion about O'. Now, suppose that when the pendulum makes an angle θ (from the downward vertical) the shadow mass is at a distance r from O'. Then $\tan\theta = r/d$, where $d = OO'$ a constant (d is the fixed distance between the pivot point of the pendulum and the horizontal plane on which the shadow moves). For small swings of the pendulum $\tan\theta \approx \theta$. Thus for small swings of the pendulum, $\theta = r/d$ and therefore, $\ddot{\theta} = \ddot{r}/d$. Substituting these in the equation of motion for the pendulum executing small swings, we obtain the equation of motion for the shadow mass as

$$\ddot{\theta} = -\theta \implies \ddot{r} = -r.$$

Thus, the equations of motion for both the pendulum and its shadow mass take the same form. Since $\ddot{r}$ is the acceleration, which is proportional to force F, the last expression can be expressed as $F = -r$ assuming that the shadow has unit mass. We have thus demonstrated how the small swings of the pendulum where the equation of motion consists of the angle θ relates to the equation of motion of its shadow which consists of the distance r. The force $F = -r$ acts as an attractive force on the shadow mass where r is the distance from the center O' of attraction.

We have therefore demonstrated how the small swings of the pendulum directly relates to an attractive force the strength of which varies linearly with the distance r from the center of attraction. Throughout the motion of a particle under such a force, the force is always directed to the center of attraction; hence it is termed a central force. These facts are encapsulated in the vectorial relation $\mathbf{F} = \ddot{\mathbf{r}} = -\mathbf{r} = -r\hat{\mathbf{r}}$ (where $\hat{\mathbf{r}}$ is the unit radial vector). Up to now we have only considered the linear simple harmonic motion about the center of attraction due to F. If the particle is given a kick perpendicular to the line joining the particle and its center of attraction under F, then the particle will move in an elliptical orbit under the same force law $F = -r$. To demonstrate this we will work in the complex plane $\mathbb{C}$ where the position of the particle is given by the complex number $z = x + iy$ where $i = \sqrt{-1}$ and $x, y \in \mathbb{R}$. The force law $\mathbf{F} = \ddot{\mathbf{r}} = -\mathbf{r}$ in the real plane $\mathbb{R}^2$ then translates into $\ddot{z} = -z$ in the complex plane $\mathbb{C}$.

To keep the analysis simple and stress on the essentials we start with a particle that moves counterclockwise on a circle of radius r with unit angular speed $\omega = 1$ rad/s in the complex plane. The movement of the particle is then given by $w = re^{i\alpha}$ where $\alpha = \omega t$ is the angle (in radians) between the real axis (i.e., the x-axis) and the line connecting the particle to the origin, and t is the time variable. Since $\omega = 1$ rad/s, we have $\alpha = t$ rad. Thus, $w = re^{it}$. We now make the transformation,

$$w \longrightarrow z = w + \frac{1}{w} = re^{it} + \frac{1}{r}e^{-it}. \tag{1}$$

The first term in the last expression corresponds to the particle that moves counterclockwise at unit angular speed on a circle of radius r as discussed above; the second term corresponds to a particle that moves clockwise on a circle of radius $1/r$ at the same unit angular speed. (The radius r needs to satisfy the condition $r > 1$, the reason for which will be described shortly.) The motion of these two circular motions superimpose to yield the resultant motion, which is an ellipse (see Figure 2).

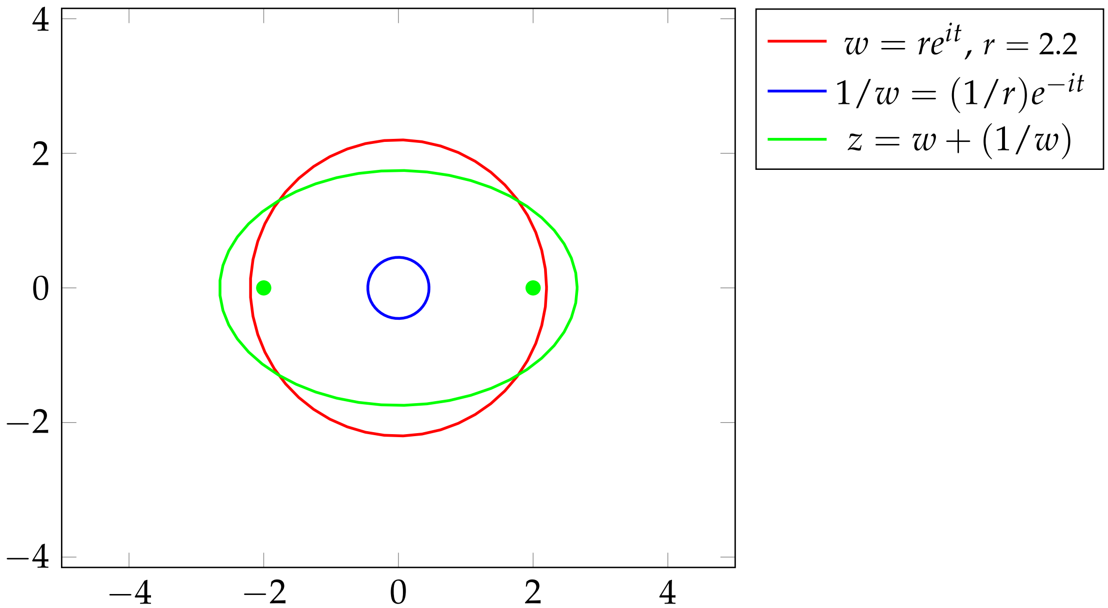


Figure 2: Two circular motions in the complex plane superimposes to yield an ellipse. The red and the blue circles, centered at the origin, are governed by $w = re^{it}$ and $\frac{1}{w} = \frac{1}{r}e^{-it}$, respectively, where $r = 2.2$. The green ellipse is tracked by $z = w + \frac{1}{w} = re^{it} + \frac{1}{r}e^{-it}$. The ellipse is centered at the origin and its two foci (green dots) are at $(\pm 2, 0)$. The ellipse described by z satisfies $\ddot{z} = -z$, which is in the form of the linear central force field $F = \ddot{r} = -r$. That is, a particle under F will move in an ellipse when a non-zero initial speed is present.

Expanding (1) using $e^{it} = \cos t + i \sin t$ and $e^{-it} = \cos t - i \sin t$, we obtain

$$z = \left(r + \frac{1}{r}\right) \cos t + i \left(r - \frac{1}{r}\right) \sin t.$$

This expression indicates that $r + \frac{1}{r} = a$ is the semi-major axis and $r - \frac{1}{r} = b$ is the semi-minor axis. z can be written in the standard complex number form $z = x + iy$ where $x = a \cos t$ and $y = b \sin t$, from which the standard form of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is evident since $\cos^2 t + \sin^2 t = 1$. Hence the foci of the ellipse are located (on the x-axis) at $\sqrt{a^2 - b^2} = \sqrt{4} = \pm 2$.

We can easily confirm that (1) satisfies the equation of motion $\ddot{z} = -z$ in the complex plane, which we remind is the analog of the central force law $\mathbf{F} = \ddot{\mathbf{r}} = -\mathbf{r}$ in the real plane. Also, $\dot{z} = i(re^{it} - \frac{1}{r}e^{-it})$. Therefore, at $t = 0$, $\dot{z}(0) = i(r - \frac{1}{r})$, which means the initial speed v with which we need to launch the particle to move on the ellipse is $v = \mid \dot{z} \mid = (r - \frac{1}{r})$. The particle needs to be launched at $r + \frac{1}{r}$ with the initial speed v in the vertical (imaginary axis) direction for it to move on an ellipse subject to the equation of motion $\ddot{z} = -z$. Since speed v is a positive value ($v > 0$), r needs to satisfy the condition $r > 1$. Thus the circle described by w from which we obtained the ellipse described by z via the mapping $z = w + \frac{1}{w}$ needs to have a radius $r > 1$.

The summary of the above discussion, then, is as follows: Starting from the equation of motion of the simple pendulum for small swings, $\ddot{\theta} = -\theta$, we established a direct link between that motion and the simple harmonic motion of the pendulum's shadow, which is governed by the equation $\ddot{r} = -r$ where the corresponding central attracting force is $\mathbf{F} = \ddot{\mathbf{r}} = -\mathbf{r}$. This force has a linear dependency on the distance from the center of attraction. The equation of motion $\ddot{r} = -r$ can be expressed in the complex plane by $\ddot{z} = -z$. The general solution to this equation takes the form (1) where, if $r > 1$, describes an ellipse with center at the origin and foci at $(\pm 2, 0)$. The center of the ellipse at the origin $(0, 0)$ is also the center of attraction, which we will denote by $\odot$. We capture this information below:

$$\boxed{\mathbf{F} = \ddot{\mathbf{r}} = -\mathbf{r} \implies \ddot{z} = -z, \;\; z = w + \frac{1}{w}, \;\; w = re^{it}, \;\; r > 1 \implies \text{ellipse, foci: } (\pm 2, 0); \;\; \odot = (0, 0).} \tag{2}$$

We now proceed to show that the ellipse described by z in the complex plane, which is associated with the force law $F \propto -r$, can be mapped to another ellipse, described by Z in the complex plane, where a particle moving on the latter ellipse is governed by the force law having the form of the inverse square law for gravitation, thereby establishing the duality between the force laws $F \propto -r$ and $\mathcal{F} \propto -\frac{1}{r^2}$.

In order to establish the stated duality between the two force fields, we recall the area law of Kepler, which arises due to the conservation of angular momentum of a particle moving in an orbit under a central force. At a given instance, if the radius vector to the particle from the center of attraction is $\mathbf{r} = r\hat{\mathbf{r}}$, then its angular momentum vector is given by,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \dot{\mathbf{r}},$$

where $\mathbf{p}$ is the momentum vector of the particle at that instance. (Here, without loss of generality, we assume the particle to have unit mass.) The rate of change of $\mathbf{L}$ is then,

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} + \mathbf{r} \times \ddot{\mathbf{r}}.$$

The first term in the last expression vanishes since it is the cross product of a vector with itself. A central force can be written in general form as $\ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}}$, where $f(r)$ is a scalar function of r.(For the central force fields F and $\mathcal{F}$ discussed above, $f(r) = -r$ and $f(r) = -\frac{1}{r^2}$, respectively.) Since $\mathbf{r} = r\hat{\mathbf{r}}$, the rate of change of $\mathbf{L}$ then becomes,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \ddot{\mathbf{r}} = r\hat{\mathbf{r}} \times f(r)\hat{\mathbf{r}} = rf(r)[\hat{\mathbf{r}} \times \hat{\mathbf{r}}] = \mathbf{0}.$$

This implies that the angular momentum vector $\mathbf{L}$ is a constant in time, and hence, the angular momentum of the particle is conserved as it traverses the orbit under a central force field. We now show that as a consequence of the conservation of angular momentum, the radial vector of the particle from the center of attraction sweeps equal areas in equal times under central force fields. This was first enunciated by Kepler in the context of planets moving about the Sun under the inverse square law of gravitation. (Kepler himself did not know about the inverse square nature of the gravitational force law at the time. These latter discoveries were due to the efforts of Hooke and Newton.)

To see the reasons behind Kepler's area law for central force fields, consider a particle moving in a central force field with the center of attraction at O (see Figure 3). Suppose the particle moves from P to Q during a small time interval Δt. Then, if the speed of the particle at P is v, the length $PQ = v\Delta t$. The altitude of the triangle OPQ is $OM = r\sin\vartheta$, where $OP = r$ and ϑ is the angle between the vectors $\mathbf{r}$ and $\mathbf{v}$. The area ΔA of the triangle OPQ is then,

$$\Delta A = \frac{1}{2}PQ \cdot OM = \frac{1}{2}(v\Delta t)(r\sin\vartheta) \implies \frac{\Delta A}{\Delta t} = \frac{1}{2}rv\sin\vartheta.$$

We recognize $rv\sin\vartheta$ to be the magnitude of the cross product between the radial and the velocity vector. Hence it is equal to $\mid \mathbf{r} \times \mathbf{v} \mid$. But $\mathbf{r} \times \mathbf{v} = \mathbf{L}$ is the angular momentum vector. Therefore, $rv\sin\vartheta = \mid \mathbf{r} \times \mathbf{v} \mid = \mid \mathbf{L} \mid$. In the limit $\Delta t \to 0$, we can then conclude that

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} \mid \mathbf{L} \mid .$$

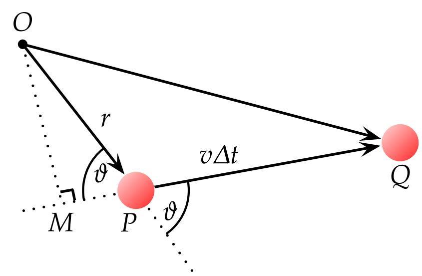


Figure 3: Diagram for establishing the area law of Kepler for central force fields. The particle moves from P to Q in a time interval Δt. The radial distance from the center of the field O to P is r. The instantaneous speed of the particle at P is v. The area ΔA of the triangle OPQ is then $\frac{1}{2}PQ \cdot OM = \frac{1}{2}(v\Delta t)(r \sin \vartheta)$, where ϑ is the angle between the radial vector $\mathbf{r} = r\hat{\mathbf{r}}$ and the velocity vector $\mathbf{v} = v\hat{\mathbf{v}}$. The radial and the velocity vectors are in the directions OP and PQ, respectively. | Drawing by AD.

We have demonstrated above that the angular momentum vector of a particle is a constant in a central force field. Therefore its magnitude $\mid \mathbf{L} \mid$ is also a constant. This implies that the rate of change of the area swept by the radial vector is also a constant.

Now, the magnitude of the angular momentum vector can be expressed as the product of the moment of inertia of the particle about the center of attraction, which is equal to r^2 (for a unit mass) and its angular speed $\frac{d\alpha}{dt}$, where α is the angle between a fixed direction and the radial vector of the particle (measured counterclockwise) at a given instance. Thus, the above expression can be recast as

$$\boxed{\frac{dA}{dt} = \frac{1}{2} \mid \mathbf{L} \mid = \frac{1}{2} r^2 \cdot \frac{d\alpha}{dt} = \text{constant. } \textbf{Area Law of Kepler}} \tag{3}$$

We now have a main ingredient to take the next step toward demonstrating the duality between the linear force law and the inverse square law of gravitation. We have already demonstrated that z describes an ellipse under the linear attractive force law $\ddot{z} = -z$. For the duality to hold, the ellipse described by z must map into another ellipse. We posit that such an ellipse is obtained by squaring z. Thus,

$$z \longrightarrow Z = z^2 = \left(w + \frac{1}{w}\right)^2 = w^2 + \frac{1}{w^2} + 2.$$

Let $w^2 = W$. Comparing the two forms, $z = w + \frac{1}{w}$ and $Z = w^2 + \frac{1}{w^2} + 2 = W + \frac{1}{W} + 2$, we note that both z and Z have the same form except that the latter is shifted by $+2$. First, the similar forms imply that Z also describes an ellipse since z describes an ellipse. This also means that the foci at $(\pm 2, 0)$ of the ellipse described by z get shifted by a $+2$ in the ellipse described by Z. Therefore, the ellipse described by Z has foci (on the x-axis) at $-2+2=0$ and at $2+2=4$. The two ellipses are shown for comparison in Figure 4.

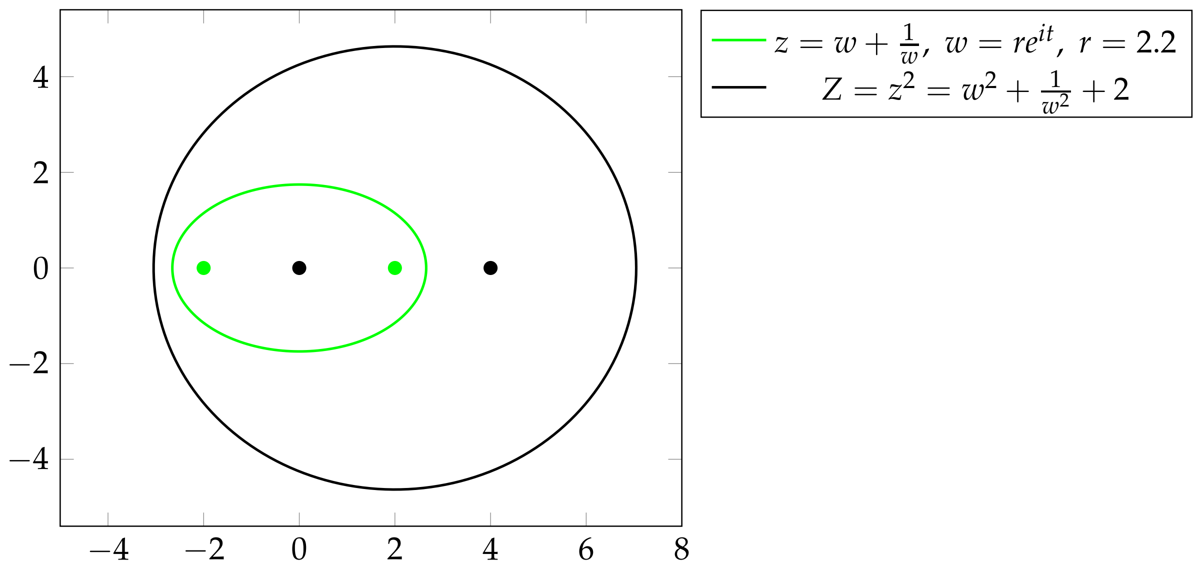
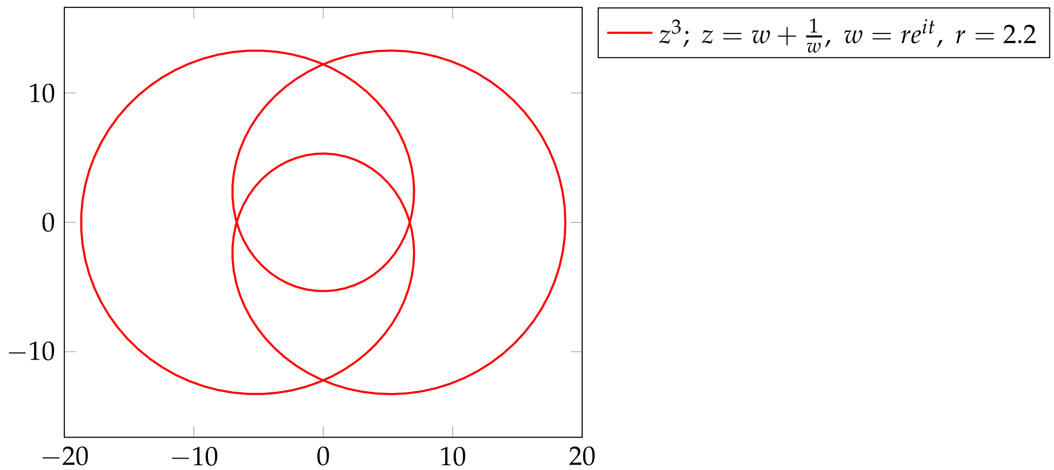


Figure 4: The ellipses described by z and $Z = z^2$. The foci (green dots) at $(\pm 2, 0)$ in the ellipse described by z are shifted by $+2$ in the ellipse described by Z (foci shown by black dots). Hence, the center of the ellipse described by z, which is also its center of attraction, becomes a focus, and hence its center of attraction for the ellipse described by Z.

Given that $z = w + \frac{1}{w}$ and $z^2 = w^2 + \frac{1}{w^2} + 2 = W + \frac{1}{W} + 2$ have the same form (except for a shift $+2$ along the real line), the mapping $z \to z^2$ is the only mapping that would take the ellipse described by z to another ellipse. To illustrate this by two other examples, Figure 5 shows the orbits for z^3 and z^4, which clearly are not ellipses. There forms have $w^3 + 3w + 3\frac{1}{w} + \frac{1}{w^3}$ and $w^4 + \frac{1}{w^4} + 4w^2 + \frac{4}{w^2} + 6$, respectively, which are clearly different from the form of z, and equivalently, z^2.



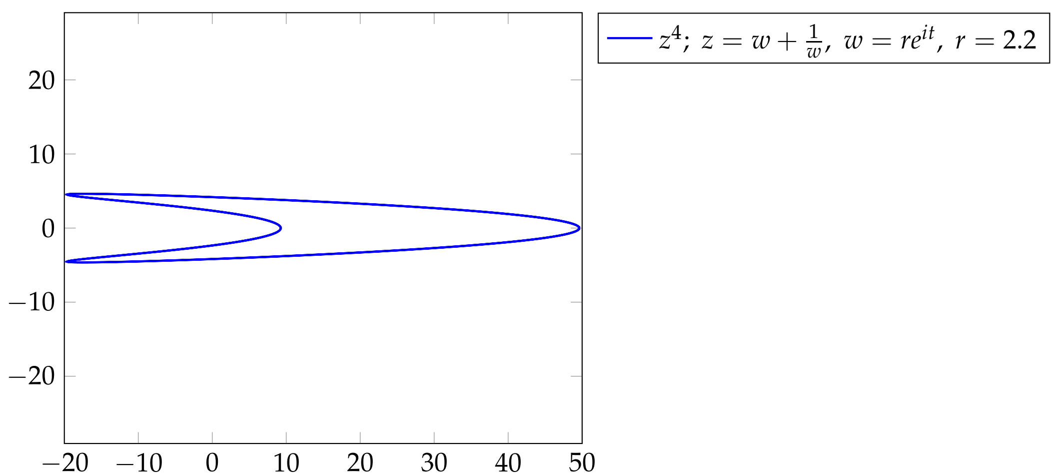


Figure 5: Orbits for z^3 and z^4, which are not elliptical. Closed elliptical orbits only occur for z^n where $n = 1, 2$.

Essentially, the center of the ellipse described by z has become a focal point of the ellipse described by Z. Another way to state this is that, in the orbit described by z, which relates to the linear force field, the attractive center of the force field is at its center; in the orbit described by Z under the mapping $z \to z^2$, the attractive center of the force field is at a focus (which is the center of attraction of the ellipse described by z). Coupling this observation with that of Kepler's first law – where planets move in elliptical orbits with the Sun at one focus – this already gives us a hint that the force field associated with the orbit described by Z must be governed by an inverse square law. We now turn to the final part of the demonstration: that is to show that the elliptical orbit described by Z is indeed governed by an inverse square law. For that we will make use of the area law derived in (3).

Since z is a general point on the ellipse described by itself, z can be written generally as $z = re^{i\alpha}$, where r is a function of α. Thus, the square of the magnitude of z at the given point on the ellipse is $\mid z \mid^2 = r^2$. Therefore, in the complex plane, (3) becomes,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\alpha}{dt} = \frac{1}{2} \mid z \mid^2 \frac{d\alpha}{dt} = \text{constant.}$$

When we obtain the ellipse described by the mapping $z \longrightarrow z^2 = Z$, then $Z = r^2 e^{i(2\alpha)}$. Thus, this mapping makes $r \to R = r^2$ and $\alpha \to \Theta = 2\alpha$. Therefore, the area $A \longrightarrow \mathcal{A}$; and since the area law must also hold in the new ellipse (assuming a central force field is at play), we have,

$$\frac{d\mathcal{A}}{dt} = \frac{1}{2}R^2\frac{d\Theta}{dt} = \frac{1}{2}(r^2)^2\frac{d(2\alpha)}{dt} = 2r^2\left(\frac{1}{2}r^2\frac{d\alpha}{dt}\right) = 2r^2\frac{dA}{dt} = 2 \mid z \mid^2 \frac{dA}{dt} \neq \text{constant.}$$

Due to the presence of the term $\mid z \mid^2$ which varies as a point traverses the ellipse described by z, we see that although $\frac{dA}{dt}$ is a constant for the ellipse described by z, $\frac{d\mathcal{A}}{dt}$ would not be a constant for the ellipse described by Z. But from the conservation of angular momentum, which equally applies to a particle moving under a central force field in the ellipse described by Z, the rate of change of $\mathcal{A}$ must be a constant. To obtain this constancy, let us redefine time t to be τ for the ellipse described by Z so that,

$$\frac{d\mathcal{A}}{dt} \to \frac{d\mathcal{A}}{d\tau} = \frac{d\mathcal{A}}{dt} \cdot \frac{dt}{d\tau} = 2 \mid z \mid^2 \frac{dA}{dt} \cdot \frac{dt}{d\tau}.$$

Let us now define,

$$\frac{dt}{d\tau} = \frac{1}{\mid z \mid^2} = \frac{1}{zz^*},$$

where z^* is the complex conjugate of z, which is obtained by letting $i \to -i$. Substituting this latest result in the above expression for $\frac{d\mathcal{A}}{dt}$, we obtain,

$$\frac{d\mathcal{A}}{d\tau} = 2 \mid z \mid^2 \frac{dA}{dt} \cdot \frac{dt}{d\tau} = 2 \mid z \mid^2 \frac{dA}{dt} \cdot \frac{1}{\mid z \mid^2} = 2\frac{dA}{dt} = \text{constant.}$$

Thus, with the introduction of the new time variable τ and defining $\frac{dt}{d\tau} = \frac{1}{|z|^2}$, we eliminate the dependency of $\frac{d\mathcal{A}}{d\tau}$ on $\mid z \mid^2$, and therefore, the area law holds for the new ellipse described by Z as it must be. We can write the expression for $\frac{dt}{d\tau}$ in the operator form as,

$$\frac{d}{d\tau} = \frac{1}{zz^*}\frac{d}{dt}.$$

Applying this operator on Z, we obtain,

$$\frac{dZ}{d\tau} = \frac{1}{zz^*}\frac{dZ}{dt}.$$

Differentiating this with respect to τ, and remembering that $Z = z^2$ and $\ddot{z} = \frac{d^2z}{dt^2} = -z$,

$$\begin{aligned}
\frac{d^2Z}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{1}{zz^*}\frac{dZ}{dt}\right) = \frac{1}{zz^*}\frac{d}{dt}\left(\frac{1}{zz^*}\frac{dZ}{dt}\right) = \frac{1}{zz^*}\frac{d}{dt}\left(\frac{1}{zz^*}\frac{dz^2}{dt}\right) = \frac{1}{zz^*}\frac{d}{dt}\left(\frac{2}{z^*}\frac{dz}{dt}\right), \\
&= \frac{2}{zz^*}\left(\frac{d}{dt}\left[\frac{1}{z^*}\right] \cdot \frac{dz}{dt} + \frac{1}{z^*}\frac{d^2z}{dt^2}\right), \\
&= -\frac{2}{zz^*}\left(\frac{1}{z^{*2}}\frac{dz^*}{dt}\frac{dz}{dt} + \frac{z}{z^*}\right) = -\frac{2}{zz^*}\left(\frac{1}{z^{*2}}\dot{z}^*\dot{z} + \frac{z}{z^*}\right) = -\frac{2}{zz^{*3}}\left(\dot{z}^*\dot{z} + z^*z\right), \\
\frac{d^2Z}{d\tau^2} &= -\frac{2}{zz^{*3}}\left(\mid \dot{z} \mid^2 + \mid z \mid^2\right).
\end{aligned}$$

The first term in the bracket can be identified as representing the square of the speed, and hence twice the kinetic energy $\mathcal{E}_k$ of the particle (of unit mass) moving on the ellipse describe by z; that is, $2\mathcal{E}_k =\mid \dot{z} \mid^2$. The second terms can be identified as twice its potential energy. This latter result can be quickly demonstrated as follows: since the central force is derivable by a potential function V that only depends on the distance, we have $F = -dV/dr = -r$. Integrating as $\int_0^{\mathcal{E}_p} dV = \int_0^r r \cdot dr \longrightarrow 2\mathcal{E}_p = r^2 \implies 2\mathcal{E}_p =\mid z \mid^2$. Therefore, the sum of the two is equal to twice its total energy $\mathcal{E}$, which from the conservation of energy, is a constant of motion. Thus,

$$\begin{aligned}
\frac{d^2Z}{d\tau^2} &= -\frac{2}{zz^{*3}}(2\mathcal{E}) = -\frac{4\mathcal{E}}{zz^{*3}}, \\
&= -4\mathcal{E}\frac{z^2}{z^3z^{*3}} = -4\mathcal{E}\frac{z^2}{(zz^*)^3} = -4\mathcal{E}\frac{z^2}{(\mid z \mid^2)^3}.
\end{aligned}$$

Since $| z^2 |=| z |^2$ and $Z = z^2$, $| Z |=| z^2 |=| z |^2$. Substituting these in the last expression above yields,

$$\frac{d^2 Z}{d\tau^2} = -4\mathcal{E}\frac{Z}{| Z |^3} \sim -\frac{\mathbf{r}}{| \mathbf{r} |^3} = -\frac{\hat{\mathbf{r}}}{r^2}. \tag{4}$$

This last expression carries the form of the inverse square law of gravitation! Thus,

$$\boxed{Z = z^2 \implies \text{ellipse, foci: } (0,0),\ (4,0);\ \odot = (0,0) \implies \mathbf{F} \sim -\frac{\mathbf{r}}{| \mathbf{r} |^3} = -\frac{\hat{\mathbf{r}}}{r^2}.} \tag{5}$$

Starting from the small swings of the pendulum, we have hence made the connection between the linear force fields and the inverse square force fields; one is dual to other. We state this beautiful connection in the form of Bohlin's Theorem:

Bohlin's Theorem

In the complex plane, a particle moving in an ellipse described by z and governed by (Hooke's) linear force law $\ddot{z} = -z$ maps to an ellipse described by Z and governed by (Newton's) inverse square gravitational law $\ddot{Z} \propto -\frac{Z}{|Z|^3}$ under $z \to Z = z^2$. In other words, if a particle moves in an ellipse under (Hooke's) linear central force law directed toward the center of the ellipse, so that $F = -r$, then, making this center of attraction a focus of a new ellipse would make the particle move along the new ellipse under (Newton's) inverse square law of gravitation $\mathcal{F} \propto -\frac{1}{r^2}$. Thus the force laws $F \propto r$ (exponent $+1$) and $\mathcal{F} \propto \frac{1}{r^2}$ (exponent -2) are dual to each other. Hence, if a central force law $F \propto r$ exists in nature, then so must the force law $\mathcal{F} \propto \frac{1}{r^2}$, and vice versa.

♣