The Spinning Water Bucket

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1 Introduction

In this essay we analyze the dynamics of a spinning water bucket. The bucket is assumed to be massless and cylindrical with radius R, which is tied to a massless rope having a torsional constant κ. The rope is tied to the ceiling and aligns with the long symmetric axis of the cylinder, which we will simply call the axis from now on. The bucket is filled with water, the surface of which is at height h from the bottom of the bucket when the entire apparatus is at rest. The mass of the water is then $m = \pi R^2 h \rho$, where ρ is the density of water. We take the system to have no dissipations, and therefore, its energy is conserved.

The bucket is then rotated by hand so that the rope twists until it makes an initial angle $\phi = \phi_0$ with respect to some reference point in space (e.g., on the walls surrounding the bucket). ϕ is the azimuthal angle measured on the plane perpendicular to the axis of the cylinder; the plane itself can be taken to coincide with the plane of the bottom of the bucket. Once the water comes to a rest after the twisting is complete, the bucket is let go such that it has only rotational motion about its axis. Our intuition (and experience) tells us that the rope will now start to unwind and the bucket's angular speed will increase as the rope untwists. When the rope has reached its original untwisted state at $\phi = 0$, the bucket will be rotating at its fastest; pass this point, the rope will start to twist in the opposite direction and the bucket will come to a rest at angle $\phi = -\phi_0$. The journey back begins... .

2 Characteristics of the Motion

There are several very interesting deductions we can make about the spinning motion of the bucket and the behavior of water contained within it. To proceed, we assume that the bucket's motion is instantaneously transferred to the water; that is, when the bucket is spinning at an angular speed ω, the water in the bucket will be spinning at this rate as well. In other words, we take the water to be rigid but nimble enough to undergo changes to its shape instantaneously in phase with the motion of the bucket itself.

2.1 Shape of the Water Surface

It is our experience that a liquid in a rotating container will adapt a concave shape. To see the exact nature of this shape in the rotating bucket let us consider a rectangular inertial frame

$\mathcal{I}$ located at the center $\mathcal{O}$ of the bottom of the bucket. We can define the inertial frame as $\mathcal{I}_0 \equiv \{\mathcal{O}, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, where $\mathbf{e}_i$ are the unit vectors attached to the frame in the directions $i = x, y, z$. We take $\mathbf{e}_z$ to be pointing along the axis (and, therefore, along the rope) of the bucket toward the ceiling.

Let us now assume that the bucket is spinning at the steady angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_z = \dot{\phi} \mathbf{e}_z$ with respect to the inertial frame and that the water surface has adapted a particular shape. Given the symmetry of the rotational motion and the shape of the bucket about its axis, we can expect the water surface to be symmetric about the axis. Now imagine a particle of mass $m_\bullet$ in equilibrium on the surface of the water located at $P = (r, z)$, where r is the distance between the particle and the axis along the line that is perpendicular to the axis. Therefore, r is the radius of the circle that the particle traces as the water surface that it sits on rotates about the axis due to the rotation of the bucket. Let us mark the center of this circle as O. We can therefore establish a rotating frame $\mathcal{R}$ such that $\mathcal{R} \equiv \{O, \mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$, where $\mathbf{e}_r$ always points toward the particle and $\mathbf{e}_\phi$ is perpendicular to $\mathbf{e}_r$ in the increasing ϕ direction. (See Figure 1.)

The velocity of the particle can then be computed with respect to an inertial frame $\mathcal{I}$ located at O (the unit vectors of which align with those of the inertial frame $\mathcal{I}_0$ at $\mathcal{O}$) by taking the time derivative of $\mathbf{r}_{P/O} = r\mathbf{e}_r$ with respect to $\mathcal{I}$. Since $\mathbf{e}_r$ changes its direction with respect to $\mathcal{I}$ as the particle rotates in a circle of constant radius r about O, we obtain

$$ {}^{\mathcal{I}}\mathbf{v}_{P/O} = \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} = r\frac{{}^{\mathcal{I}}d}{dt}\mathbf{e}_r = r\left(\frac{{}^{\mathcal{R}}d}{dt}\mathbf{e}_r + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{e}_r\right) = r\left(\mathbf{0} + \omega\mathbf{e}_z \times \mathbf{e}_r\right) = r\omega\mathbf{e}_\phi, \tag{1} $$

where ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$ is the angular velocity vector of the particle (bucket), which is equivalent to $\omega\mathbf{e}_z$. (The rate of change of the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$, and $\mathbf{e}_z$ with respect to the rotating frame $\mathcal{R}$ vanishes since these vectors are attached to it.) Differentiating the above expression again with respect to time in the inertial frame $\mathcal{I}$, we obtain the acceleration vector of the particle as,

$$ {}^{\mathcal{I}}\mathbf{a}_{P/O} = \frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\mathbf{v}_{P/O} = r\omega\frac{{}^{\mathcal{I}}d}{dt}\left(\mathbf{e}_\phi\right) = -r\omega^2\mathbf{e}_r, \tag{2} $$

where we have used

$$ \frac{{}^{\mathcal{I}}d}{dt}\mathbf{e}_\phi = \left(\frac{{}^{\mathcal{R}}d}{dt}\mathbf{e}_\phi + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{e}_\phi\right) = \left(\mathbf{0} + \omega\mathbf{e}_z \times \mathbf{e}_\phi\right) = -\omega\mathbf{e}_r. $$

Newton's second law applied to the particle with respect to the rotating frame is then,

$$ m_\bullet\, {}^{\mathcal{R}}\mathbf{a}_{P/O} = \mathbf{F}_P - m_\bullet\, {}^{\mathcal{I}}\mathbf{a}_{P/O}, \tag{3} $$

where $\mathbf{F}_P$ is the net force on the particle. The forces on the particle can be taken to consist of the force of gravity $\mathbf{F}_G = -m_\bullet g\mathbf{e}_z$ and the force $\mathbf{F}_N$ exerted by the water, which would be normal to the tangent of the water surface at P; g is the constant gravitational acceleration. For the particle

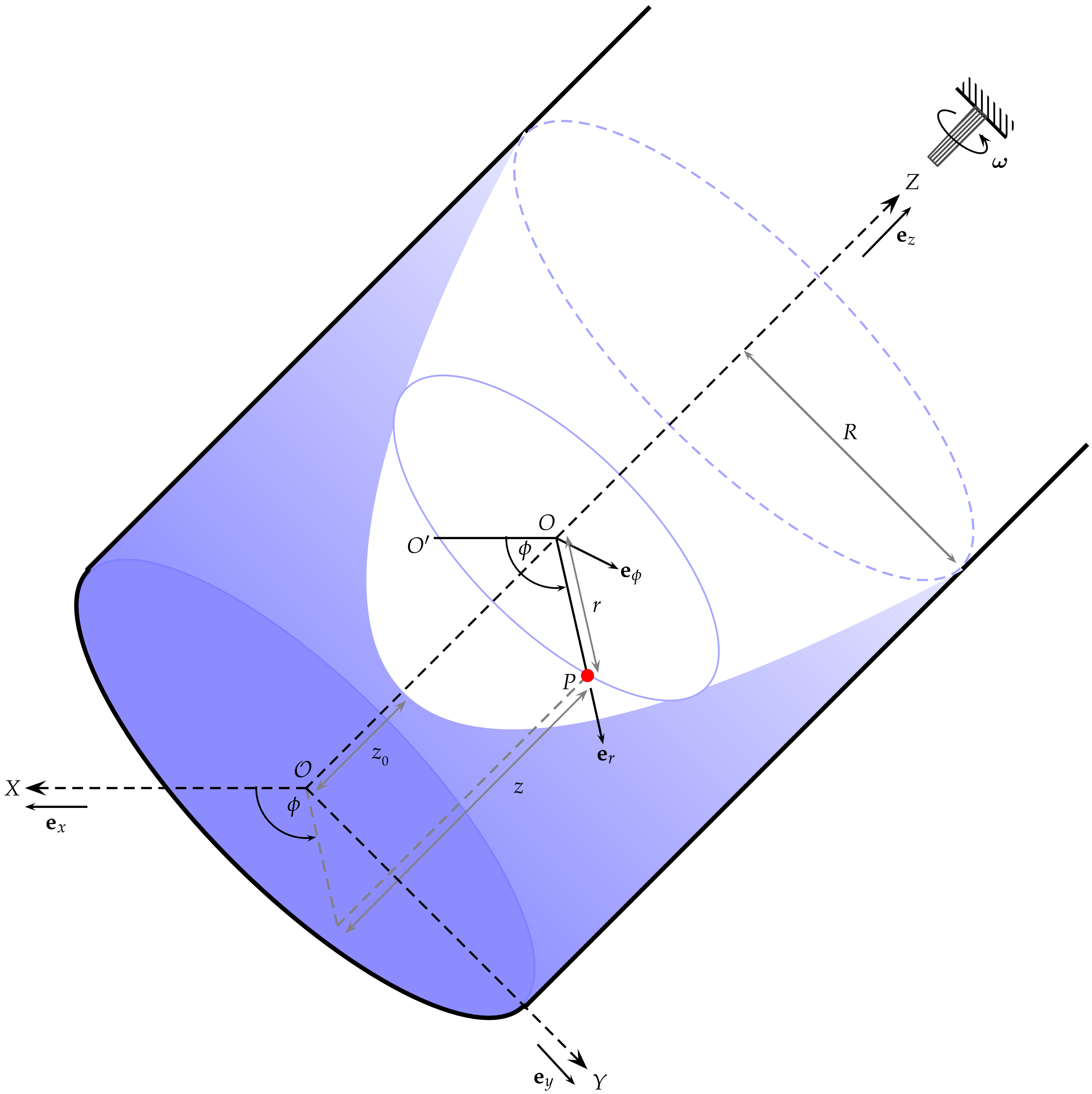

Figure 1: The schematic of the cylindrical water bucket of radius R spinning at angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_z$. The inertial frame is $\mathcal{I}_0 \equiv \{\mathcal{O}, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, and the rotating frame of the point $P = (r, z)$ is $\mathcal{R} \equiv \{O, \mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$. Note that $\mathbf{e}_\phi \perp \mathbf{e}_r$, the reference line OO' is parallel to the X-axis, and the dashed line that projects from P to the bottom of the bucket is perpendicular the bottom plane of the bucket. The diagram is drawn at an angle to depict the details although the Z-axis is vertical for the system considered. | Drawing by AD.

to exist in equilibrium on the water surface (in the rotating frame), $\mathbf{F}_N$ will have to point away from the water. If the tangent to the water surface makes an angle ϑ with the downward vertical at P, then $\mathbf{F}_N = -F_N \cos\vartheta \mathbf{e}_r + F_N \sin\vartheta \mathbf{e}_z$. (See Figure 2.)

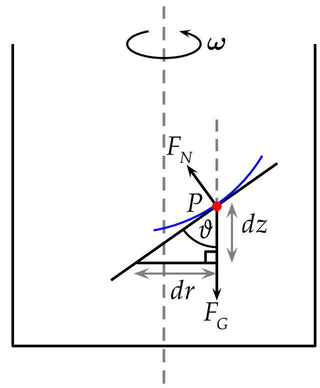

Figure 2: Two-dimensional cross-sectional view of the forces acting on a particle (red dot) on the water surface (blue curve) contained in a cylindrical bucket spinning at constant angular velocity $\boldsymbol{\omega}$ about its symmetric axis. The particle, located at P, moves around the axis of the cylinder in a circle with respect to an inertial frame fixed on the axis; it is in equilibrium with respect to a frame rotating at the same angular velocity as the bucket. The tangent to the water surface at P forms an angle ϑ to the downward vertical at P. F_N is the force normal to the tangent at P acting on the particle; F_G is the gravitational force acting vertically downward on the particle. | Drawing by AD.

Thus,

$$\mathbf{F}_P = \mathbf{F}_N + \mathbf{F}_G = -F_N \cos\vartheta \mathbf{e}_r + F_N \sin\vartheta \mathbf{e}_z - m_\bullet g \mathbf{e}_z.$$

Substituting these expressions in (3), we obtain,

$$m_\bullet\, {}^{\mathcal{R}}\mathbf{a}_{P/O} = (m_\bullet r\omega^2 - F_N \cos\vartheta)\mathbf{e}_r + (F_N \sin\vartheta - m_\bullet g)\mathbf{e}_z.$$

Since the particle is at rest with respect to the rotating frame, its corresponding acceleration vanishes: thus, ${}^{\mathcal{R}}\mathbf{a}_{P/O} = \mathbf{0}$. Accordingly, each of the vector components on the right hand side of the last expression must vanish. This yields

$$F_N \sin\vartheta = m_\bullet g, \quad F_N \cos\vartheta = m_\bullet r\omega^2,$$

from which we obtain

$$\tan\vartheta = \frac{g}{r\omega^2}.$$

Now, $\tan\vartheta = \frac{dr}{dz}$. Thus,

$$\frac{dr}{dz} = \frac{g}{r\omega^2} \implies dz = \frac{\omega^2}{g} r \cdot dr.$$

If the water surface meets the axis at a height z_0, then $z = z_0$ at $r = 0$. Thus,

$$\int_{z_0}^{z} dz = \frac{\omega^2}{g} \int_0^r r \cdot dr,$$

which finally results in

$$\boxed{z = z_0 + \frac{\omega^2}{2g} \cdot r^2. \quad \text{Shape of the Water Surface: Parabola}} \tag{4}$$

This, then, expresses the water height from the bottom of the container and we immediately recognize that the water surface takes the shape of a parabola when the bucket is spinning. Note that in deriving (4) there was no explicit place where the bucket's cylindrical shape was taken into account. In other words, the liquid contained will form a parabolic shape when the container is rotated about its symmetric axis regardless of its shape. Also, given the symmetry of the system, the water height is independent of the azimuthal angle ϕ. These conclusions agree with our experience.

The water height, z_R, at the walls of the cylindrical bucket is now obtained by putting $r = R$. Thus,

$$z_R = z_0 + \frac{\omega^2 R^2}{2g}. \tag{5}$$

2.2 Constraints on the Angular Speed and the Initial Water Height

We recognize that z_0, which is the height at which the water meets the axis, is not a constant; it depends on the angular speed ω with which the bucket is spinning. z_0 is certainly h when the bucket is at rest, and our experience tells us that it decreases as the angular speed of the bucket increases. Similarly, z_R is h when the bucket is at rest, and it increases as the angular speed of the bucket increases. In other words, as the angular speed of the bucket increases, the water on the axis sinks toward the bottom of the bucket and the water that hugs the bucket wall rises. Let us now investigate the dependency of z_0 on the angular speed of the bucket.

At an angular speed ω, the water mark on the walls is at a height z_R given by (5). Let us imagine a cylindrical water volume of height z_R, which is simply, $V_{\text{cyl}} = \pi R^2 \cdot z_R$. Now consider the volume of the water-less paraboloid that forms when the bucket is spinning at angular speed ω; this paraboloid meets the wall of the bucket at z_R. An infinitesimal disk making up this paraboloid at point $P = (r, z)$ has volume $dV_{\text{prb}} = \pi r^2 \cdot dz$. From (4), it follows that $dz = (\omega^2/g) r \cdot dr$. Thus, $dV_{\text{prb}} = (\pi\omega^2/g) r^3 \cdot dr$. Integrating this expression from $r = 0$ to $r = R$ yields, $V_{\text{prb}} = [\pi\omega^2/(4g)] R^4$. The actual water volume is therefore given by the difference $V_{\text{cyl}} - V_{\text{prb}}$, which is equivalent to the water volume when the system is at rest; that is, to $\pi R^2 h$. (Here we assume that no water has spilled out of the bucket due to rotation.) Thus,

$$\pi R^2 h = \pi R^2 \cdot z_R - \frac{\pi\omega^2}{4g} R^4 = \pi R^2 (z_0 + \frac{\omega^2}{2g} R^2) - \frac{\pi\omega^2}{4g} R^4 = \pi R^2 z_0 + \frac{\pi\omega^2}{4g} R^4.$$

Rearranging, we obtain,

$$\boxed{z_0 = h - \frac{\omega^2}{4g} R^2.} \tag{6}$$

The only variable on the right hand side (6) is the angular speed ω. Hence, as expected, as the angular speed increases, the height at which the water surface meets the axis decreases (with the decrease varying as angular speed squared). To see this more clearly, given that the last term on the right is always ≥ 0, this implies that $h - z_0 \geq 0$, which in turn implies that $z_0 \leq h$. As also expected, $z_0 = h$ when the bucket is at rest.

With the expression (6) we are able to arrive at the maximum allowable angular speed at which $z_0 = 0$. This is the scenario where the lowest point of the parabolic water surface touches the bottom of the bucket. Let this maximum angular speed be denoted by Ω. Then from (6), it follows that

$$\boxed{z_0 = 0 \implies |\Omega| = \frac{2}{R}\sqrt{gh}. \quad \text{Maximum Allowed Angular Speed}} \tag{7}$$

Since $R^2/(4g) = h/\Omega^2$, (6) can be written as,

$$\boxed{z_0 = h\left(1 - \frac{\omega^2}{\Omega^2}\right), \quad 0 \leq z_0 \leq h.} \tag{8}$$

Using the above expressions in (4) yields,

$$\boxed{\frac{z}{h} = 1 - \frac{\omega^2}{\Omega^2}\left(1 - 2\frac{r^2}{R^2}\right). \quad \text{Fractional Water Height}} \tag{9}$$

Equation (9) allows us to find the height of the water surface at any angular speed ω at any radial distance r from the axis. What is the water height at the walls of the bucket when it is spinning at maximum angular speed Ω? Using $\omega = \Omega$ and $r = R$ in (9) reveals that $z_{\max} = 2h$. Thus, when the bucket is spinning at maximum angular speed Ω (so that the paraboloid formed by the water surface touches the bottom of the bucket), the water rises to twice its original height at the walls. So, if a liquid is not allowed to spill out under rotations of a cylindrical bucket of height H, it must not be filled more than its halfway mark at $H/2$. We summarize these findings in the box below:

$$\boxed{\frac{z}{h} = 1 - \frac{\omega^2}{\Omega^2}\left(1 - 2\frac{r^2}{R^2}\right), \quad -\Omega \le \omega \le \Omega = \frac{2}{R}\sqrt{gh}, \quad -R \le r \le R, \quad 0 \le z \le 2h.} \tag{10}$$

There are symmetries that are reflected in (10). For example, we see that the water height z is a function of ω^2, so it remains the same under $\omega \to -\omega$. This means that by looking at the water height alone we will not be able to distinguish whether the bucket is rotating clockwise or counterclockwise. The symmetry of the system about the axis along with the isotropy of space leads to this symmetry. Similarly, the water height is a function of r^2, so it remains the same under $r \to -r$. The symmetry of the system about the axis along with the homogeneity of space leads to this symmetry. Another way of saying this is, that knowing these symmetries must hold for the system of the spinning water bucket, we could have guessed that the water height must depend on even powers of ω and r. Figures 3 and 4 visualize the variations determined by (10).

2.3 The Moment of Inertia of Water

We now embark to compute the moment of inertia of water when the bucket is spinning at angular speed ω. For this, let us first compute the moment of inertia of a disk about an axis that passes through its center. Let us assume that the disk has radius R and surface density σ. Taking the disk to have negligible thickness, the mass of the disk is then, $m_{\text{disk}} = \pi R^2 \sigma$. Now, consider a small patch of area $dA = (r \cdot d\theta) \cdot dr$ on the disk located at a distance r from its center. This patch then has mass $dm = \sigma \cdot dA$. Thus, the moment of inertia of the disk about an axis that passes through its center is:

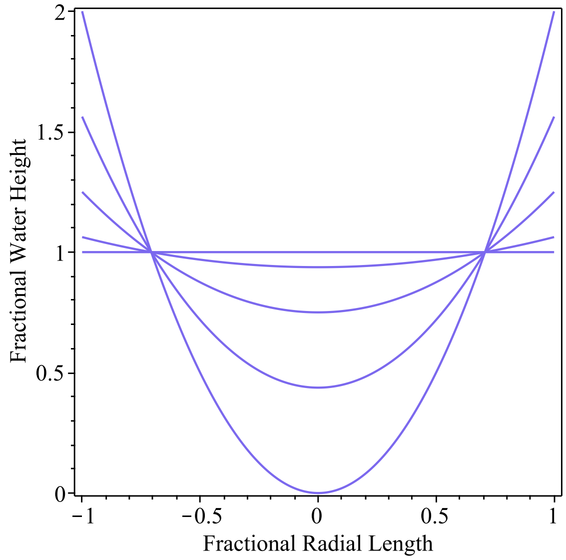

Figure 3: Fractional water height (z/h) against the fractional radial length (r/R) for different fractional angular speeds $\omega = |\frac{\omega}{\Omega}|$. In decreasing order of $z/h \leq 1$: $\omega = 0$ (flat water surface; bucket is at rest), $\omega = 1/4$, $\omega = 1/2$, $\omega = 3/4$, $\omega = 1$ (water surface touches the bottom of the bucket; bucket is spinning maximally without spilling water). These curves represent the (two-dimensional cross-sectional) shape of the water surface under the different conditions stated.

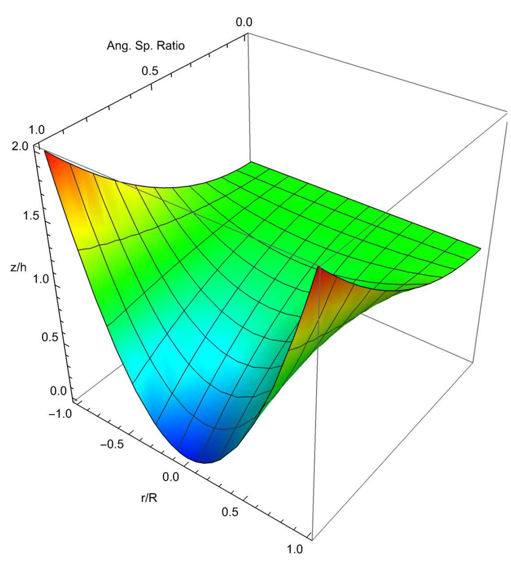

Figure 4: Variation of the fractional water height (z/h) against the fractional radial length (r/R) for different fractional angular speeds $\omega = |\frac{\omega}{\Omega}|$ (Ang. Sp. Ratio).

$$I_{\text{dsk}} = \int (dm)r^2 = \sigma \int_0^R r^3 \cdot dr \int_0^{2\pi} d\theta = \frac{1}{2}(\pi R^2 \sigma)R^2 = \frac{1}{2} m_{\text{dsk}} R^2.$$

We can use the result for I_{dsk} to find the moment of a cylinder of radius R and mass m_{cyl} about its long axis since the cylinder can be considered to be made up of an infinite number of thin stacked disks each of radius R. Thus,

$$I_{\text{cyl}} = \sum_i \frac{1}{2}(m_{\text{dsk}})_i R^2 = \frac{1}{2}R^2 \cdot \sum_i (m_{\text{dsk}})_i = \frac{1}{2} m_{\text{cyl}} R^2.$$

Returning to the spinning water bucket, let us assume that it is rotating at steady angular speed ω. Putting $r = R$ in (10), the height at which the water surface hugs the wall of the cylindrical bucket is,

$$z_R = h(1 + \text{w}^2), \quad \text{w} = \frac{\omega}{\Omega}.$$

As before, let us imagine a cylinder of radius R and height z_R filled with water of density ρ. The moment of inertia of this water cylinder is then, $I_{\text{cyl}} = \frac{1}{2} m_{\text{cyl}} R^2$. But $m_{\text{cyl}} = \pi R^2 z_R \rho = \pi R^2 h \rho (1 + \text{w}^2)$. Now, $\pi R^2 h \rho = m$ is the mass of the water (recall that when the system is at rest, the flat water surface is at a height h). Therefore, $m_{\text{cyl}} = m(1 + \text{w}^2)$. Collecting all these terms, we conclude that the moment of inertia of the cylindrical water mass about its axis is,

$$I_{\text{cyl}} = \frac{1}{2} m R^2 \cdot (1 + \text{w}^2). \tag{11}$$

We now consider the paraboloidal shape formed by the water surface due to the spin of the bucket. Although this paraboloid is empty in content, let us assume that it is filled with water. Note that this water paraboloid is a sub-volume of the cylindrical water volume we just analyzed. The paraboloid can be regarded as made up of disks of varying radii. Consider a disk of radius r with an infinitesimal mass dm_{dsk}. The moment of inertia of the paraboloid about its (symmetric) axis is then,

$$I_{\text{prb}} = \int \frac{1}{2}(dm_{\text{dsk}})r^2.$$

If the infinitesimal disk has thickness dz, then, $dm_{\text{dsk}} = \pi r^2 (dz) \rho$. From (10), it follows that $dz = (4h\text{w}^2/R^2) r \cdot dr$. Using these expressions in the integrand above and integrating it from $r = 0$ to $r = R$, we obtain,

$$I_{\text{prb}} = \frac{1}{2} m R^2 \cdot \frac{2}{3} \text{w}^2, \tag{12}$$

where, again, $m = \pi R^2 h\rho$ is the water mass in the bucket. The moment of inertia of water in the bucket about its axis is then obtained by subtracting I_{prb} from I_{cyl}. Noting that $(1/2)mR^2 = I_0$ is the moment of inertia of water about the axis when the system is at rest, we finally obtain,

$$\boxed{I = I_0\left(1 + \frac{1}{3}\frac{\omega^2}{\Omega^2}\right), \quad I_0 = \frac{1}{2}mR^2, \quad -\Omega \le \omega \le \Omega = \frac{2}{R}\sqrt{gh}, \quad I_0 \le I \le \frac{4}{3}I_0, \quad m = \text{water mass.}} \tag{13}$$

We therefore have deduced, that as the angular speed of the bucket increases, the moment of inertia of water increases up to a maximum of four-thirds of its moment of inertia value at rest. This increase in the moment of inertia is understandable, since, as the angular speed of the bucket increases, more water will move away from the axis toward the wall of the bucket (see Figure 3). We also note that, since $I_0 = \frac{1}{2}mR^2$ and $\Omega^2 = 4gh/R^2$,

$$I_0\Omega^2 = 2mgh. \tag{14}$$

In the next section, we will show that the potential energy of the water when it is at rest (hence, when it forms a flat surface at height h) is $mgh/2$. Thus, $I_0\Omega^2$ corresponds to four times the potential energy of the water at rest.

2.4 Energy of the System

As we have stated earlier, our bucket is massless, and therefore, the entire energy of the system is carried by the water contained in it and the rope. Let us first compute the potential energy of the water. For this, just as with calculations done for determining the moment of inertia, we first consider a cylindrical water column of height $z_R = h(1 + \omega^2)$, where $\omega = \omega/\Omega$. Considering a water disk of radius R of infinitesimal height dz located at a height z, its infinitesimal potential energy is: $d\mathcal{E}_{p[\text{cyl}]} = (dm_{\text{cyl}})gz = (\pi R^2 \cdot dz \cdot \rho)gz$ (where we have taken the zero potential energy level to be the bottom of the bucket). With $m = \pi R^2 h\rho$ being the water mass, the total potential energy of the cylindrical water column is,

$$\mathcal{E}_{p[\text{cyl}]} = (\pi R^2 \rho g)\int_0^{z_R} z \cdot dz = \frac{1}{2}(\pi R^2 \rho g) z_R^2 = \frac{1}{2}(\pi R^2 h\rho) gh(1+\omega^2)^2 = \frac{1}{2}mgh(1+\omega^2)^2. \tag{15}$$

Similarly, we now compute the potential energy of the paraboloid which forms part of the water cylinder. For this, just as in the corresponding computation of moment of inertia, we consider a water disk of radius r at a height z. The infinitesimal potential energy of the paraboloid is then $d\mathcal{E}_{p[\text{prb}]} = (dm_{\text{prb}})gz = (\pi r^2 \cdot dz \cdot \rho)gz$. Using the expression for z in (10) and its differential, which gives $dz = (4h\omega^2/R^2)r \cdot dr$, the infinitesimal potential energy of the paraboloid takes the form: $d\mathcal{E}_{p[\text{prb}]} = (4\pi\rho g h^2\omega^2/R^2)[(1-\omega^2)r^3 + (2\omega^2/R^2)r^5]dr$. Integrating this expression from $r = 0$ to $r = R$ and simplifying, we obtain,

$$\mathcal{E}_{p[\text{prb}]} = \frac{4\pi\rho g h^2\omega^2}{R^2}\int_0^R \left[(1-\omega^2)r^3 + \frac{2\omega^2}{R^2}r^5\right] dr = mgh\omega^2\left(1+\frac{1}{3}\omega^2\right). \tag{16}$$

The potential energy of the water contained in the bucket when its angular speed is ω is then obtained by subtracting $\mathcal{E}_{p[\text{prb}]}$ from $\mathcal{E}_{p[\text{cyl}]}$, with the result being

$$\boxed{\mathcal{E}_{p[\text{w}]} = \frac{1}{2}mgh\left(1+\frac{1}{3}\frac{\omega^4}{\Omega^4}\right), \quad |\omega| \le \Omega = \frac{2}{R}\sqrt{gh}, \quad \text{Potential Energy of Water;} \quad m = \text{water mass.}} \tag{17}$$

Thus, the potential energy of the water grows as the fourth power of the angular speed of the bucket. When the system is at rest the potential energy of the water is $mgh/2$.

The kinetic energy of the water due to rotation is simply

$$\boxed{\mathcal{E}_{k[\text{w}]} = \frac{1}{2}I\omega^2, \quad I = I_0\left(1+\frac{1}{3}\frac{\omega^2}{\Omega^2}\right), \quad |\omega| \le \Omega = \frac{2}{R}\sqrt{gh}, \quad \text{Kinetic Energy of Water}} \tag{18}$$

where the expression for I has been proved earlier. When the rope is twisted by an angle ϕ about its axis from its unwounded position, it carries a potential energy of

$$\boxed{\mathcal{E}_{p[\text{r}]} = \frac{1}{2}\kappa\phi^2, \quad \text{Potential Energy of the Rope}} \tag{19}$$

where κ is the torsional constant of the rope. This expression is akin to the energy stored in a compressed or a stretch spring. Since the rope has been twisted by an angle ϕ_0 initially, it then has $\frac{1}{2}\kappa\phi_0^2$ amount of potential energy to begin with. Thus, at $t = 0$, the total initial potential energy of the system (water+rope) is,

$$\mathcal{E}_p(0) = \mathcal{E}_{p[\mathrm{w}]}(0) + \mathcal{E}_{p[\mathrm{r}]}(0) = \frac{1}{2}mgh + \frac{1}{2}\kappa\phi_0^2.$$

Since the bucket is released from rest, there is no kinetic energy in the system initially. Thus, $\mathcal{E}_k(0) = 0$. The total initial energy of the system is then,

$$\mathcal{E}(0) = \mathcal{E}_k(0) + \mathcal{E}_p(0) = 0 + \frac{1}{2}mgh + \frac{1}{2}\kappa\phi_0^2. \tag{20}$$

When the bucket has untwisted for some time $t = t$, and the rope is making a twist angle ϕ, the system's total energy is given by $\mathcal{E}(t) = \mathcal{E}_k(t) + \mathcal{E}_p(t)$, where $\mathcal{E}_k(t)$ is given by (18) and $\mathcal{E}_p(t)$ is the sum of the expressions (17) and (19). Therefore,

$$\boxed{\mathcal{E}(t) = \frac{1}{2}I_0\left(1 + \frac{1}{3}\frac{\omega^2}{\Omega^2}\right)\omega^2 + \frac{1}{2}mgh\left(1 + \frac{1}{3}\frac{\omega^4}{\Omega^4}\right) + \frac{1}{2}\kappa\phi^2. \quad \text{Total Energy at instant } t.} \tag{21}$$

With these results in place, we can now investigate the velocity and the acceleration of the system.

2.5 Velocity & Acceleration

The two expressions (20) and (21) are equal since the energy of the system is conserved. Hence,

$$\frac{1}{2}mgh + \frac{1}{2}\kappa\phi_0^2 = \frac{1}{2}I_0\left(1 + \frac{1}{3}\frac{\omega^2}{\Omega^2}\right)\omega^2 + \frac{1}{2}mgh\left(1 + \frac{1}{3}\frac{\omega^4}{\Omega^4}\right) + \frac{1}{2}\kappa\phi^2.$$

Using the fact that $I_0\Omega^2 = 2mgh$ [see (14)], we can simplify this expression to read

$$\varpi^4 + 2\varpi^2 - \alpha\left(1 - \frac{\phi^2}{\phi_0^2}\right) = 0, \tag{22}$$

where $\varpi = \frac{\omega}{\Omega}$ is the fractional angular velocity and

$$\alpha = \frac{\kappa\phi_0^2}{mgh}, \quad \alpha \geq 0. \tag{23}$$

The constant α will play an important role going forward. Note that it is the ratio of the initial torsional potential energy stored in the rope ($\kappa\phi_0^2/2$) to the initial potential energy of the water ($mgh/2$). Noting that (22) is a quadratic equation in ϖ^2, its two solutions are,

$$\varpi^2 = -1 \pm \left[1 + \alpha\left(1 - \frac{\phi^2}{\phi_0^2}\right)\right]^{1/2}.$$

Since $\omega^2 \geq 0$ we will have to rule out the negative solution. Thus, we conclude that,

$$\omega^2 = \frac{\omega^2}{\Omega^2} = \left[1 + \alpha\left(1 - \frac{\phi^2}{\phi_0^2}\right)\right]^{1/2} - 1, \tag{24}$$

and therefore,

$$\omega = \dot{\phi} = \pm\Omega\left\{\left[1 + \alpha\left(1 - \frac{\phi^2}{\phi_0^2}\right)\right]^{1/2} - 1\right\}^{1/2}, \quad \alpha = \frac{\kappa\phi_0^2}{mgh}, \quad \alpha \geq 0. \tag{25}$$

It is clear that during rotation, for a given set of initial conditions, $\omega = \dot{\phi}$ reaches a maximum magnitude when $\phi = 0$; that is, at the instant when the rope is fully unwound. But we also know that $|\omega|$ cannot exceed Ω since, otherwise, it will lead to water spilling out of the bucket (assuming that the water was initially filled to the halfway mark of the bucket, which is the maximum allowed rest water height as we proved earlier). Therefore, allowing $|\omega| = \Omega$ at $\phi = 0$ in (25), we establish the condition,

$$\Omega = \Omega\left\{\left[1 + \alpha\left(1 - \frac{0}{\phi_0^2}\right)\right]^{1/2} - 1\right\}^{1/2} \implies 1 = (1+\alpha)^{1/2} - 1 \implies \alpha = 3.$$

Thus, if the water is to not spill out of the bucket, we will have to contain the maximum angular speed of the bucket to Ω when the rope has fully unwound (at $\phi = 0$), which constrains α to a maximum value of 3.

$$\boxed{\alpha_{\max} = 3. \text{ No Spill Condition}} \tag{26}$$

Hence, for water to not spill out, the initial torsional potential energy of the rope must not exceed three times that of the potential energy of the water at rest. We summarize the results in the box below:

$$\boxed{\omega = \dot{\phi} = \pm\Omega\left\{\left[1 + \alpha\left(1 - \frac{\phi^2}{\phi_0^2}\right)\right]^{1/2} - 1\right\}^{1/2}, \quad |\omega| \leq \Omega = \frac{2}{R}\sqrt{gh}, \quad \alpha = \frac{\kappa\phi_0^2}{mgh}, \quad 0 \leq \alpha \leq 3.} \tag{27}$$

As can be deduced from (27),

$$\dot{\phi}|_{\phi=\phi_0} = 0 = \dot{\phi}_{\min}, \quad \dot{\phi}|_{\phi=0} = \pm\Omega\left[\sqrt{1+\alpha}-1\right]^{1/2} = \dot{\phi}_{\max}.$$

A positive (negative) sign in (27) corresponds to counterclockwise (clockwise) twist/rotations of the rope/bucket. Figure 5 shows the fractional angular speed ($\omega = |\omega/\Omega|$) as a function of the fractional angle (ϕ/ϕ_0) for several values of α. As is clear, regardless of the α value, the angular speed (hence, the ratio $|\omega/\Omega|$) is maximized when the rope is fully unwound at $\phi = 0$ (hence, when $\phi/\phi_0 = 0$). These conclusions can, of course, also be derived by computing the first and the second derivatives of $\dot{\phi}$ with respect to ϕ.

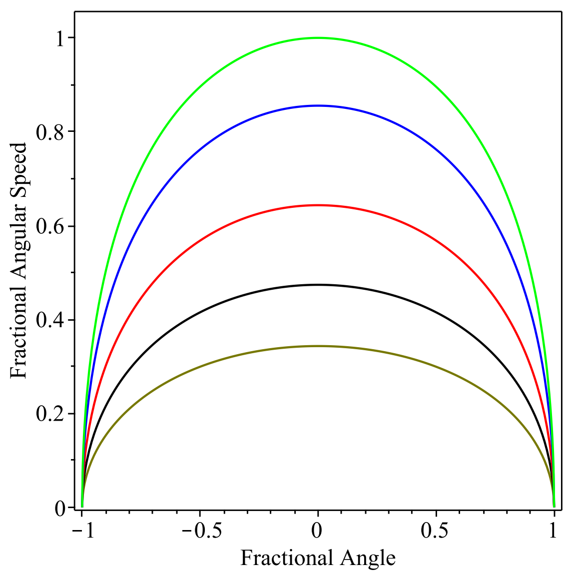

Figure 5: The fractional angular speed of the water/bucket ($\omega = |\omega/\Omega|$; vertical axis) as a function of the fractional angle (ϕ/ϕ_0; horizontal axis) for several values of α: $\alpha = 0.25$ (olive), $\alpha = 0.5$ (black), $\alpha = 1$ (red), $\alpha = 2$ (blue), $\alpha = 3$ (green).

We can differentiate (27) with respect to time to obtain the angular acceleration of the system, which reads (after substituting (27) back into the resulting intermediate expression),

$$\boxed{\ddot{\phi} = \mp\frac{\alpha\Omega^2}{2\phi_0^2}\cdot\phi\cdot\left[1+\alpha\left(1-\frac{\phi^2}{\phi_0^2}\right)\right]^{-1/2}, \quad |\omega| \leq \Omega = \frac{2}{R}\sqrt{gh}, \quad \alpha = \frac{\kappa\phi_0^2}{mgh}, \quad 0 \leq \alpha \leq 3.} \qquad (28)$$

The signs in front (27) and (28) reveal that the angular velocity and angular acceleration are in opposite directions. This opposing effect is what slows down the rotation of the bucket when it is spinning in a particular direction (clockwise or counterclockwise). When the rope has completely unwound at $\phi = 0$, we see that the angular acceleration is zero. It reaches a maximum at the maximum twist when $\phi = \phi_0$ at which point it is equal to $\mp\frac{\alpha\Omega^2}{2\phi_0}$. In summary,

$$\ddot{\phi}|_{\phi=0} = 0 = \ddot{\phi}_{\min}, \quad \ddot{\phi}|_{\phi=\phi_0} = \mp\frac{\alpha\Omega^2}{2\phi_0} = \dot{\phi}_{\max}.$$

We will now work toward deducing the inertial velocity of the bucket wall. The radial vector of a point on the bucket wall is $\mathbf{R}_{B/O} = R\mathbf{e}_r$, where B stands for the bucket. Now, R is a constant but the radial unit vector $\mathbf{e}_r$ changes with respect to the inertial frame at O as the point on the bucket wall moves in a circle around it. Hence, ${}^{\mathcal{I}}\mathbf{v}_{B/O} = \mathbf{v}_B = \frac{{}^{\mathcal{I}}d}{dt}\mathbf{R}_{B/O} = R\frac{{}^{\mathcal{I}}d}{dt}\mathbf{e}_r = R\omega\mathbf{e}_\phi$. Thus,

$$\boxed{\mathbf{v}_B = R\dot{\phi}\mathbf{e}_\phi, \quad \dot{\phi} = \pm\Omega\left\{\left[1+\alpha\left(1-\frac{\phi^2}{\phi_0^2}\right)\right]^{1/2} - 1\right\}^{1/2}, \quad 0 \le \alpha \le 3.} \tag{29}$$

Thus the velocity of the bucket wall can be considered to be represented by the same curves as in Figure 5 scaled by the factor R. The inertial acceleration of the bucket wall is then ${}^{\mathcal{I}}\mathbf{a}_{B/O} = \mathbf{a}_B = \frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\mathbf{v}_{B/O} = \frac{{}^{\mathcal{I}}d}{dt}(R\omega\mathbf{e}_\phi) = R\omega\frac{{}^{\mathcal{I}}d}{dt}\mathbf{e}_\phi + R(\frac{d}{dt}\omega)\mathbf{e}_\phi = -R\omega^2\mathbf{e}_r + R\dot{\omega}\mathbf{e}_\phi$. Therefore,

$$\boxed{\mathbf{a}_B = -R\dot{\phi}^2\mathbf{e}_r + R\ddot{\phi}\mathbf{e}_\phi, \quad \ddot{\phi} = \mp\frac{\alpha\Omega^2}{2\phi_0^2}\cdot\phi\cdot\left[1+\alpha\left(1-\frac{\phi^2}{\phi_0^2}\right)\right]^{-1/2}, \quad 0 \le \alpha \le 3.} \tag{30}$$

The square magnitude of the inertial acceleration of the bucket wall is thus, $a_B^2 = R^2(\dot{\phi}^4 + \ddot{\phi}^2)$, which, using (27) and (28) can be written as,

$$\boxed{\frac{a_B^2}{R^2\Omega^4} = \left\{\left[1+\alpha\left(1-\frac{\phi^2}{\phi_0^2}\right)\right]^{1/2} - 1\right\}^2 + \frac{\alpha^2\phi^2}{4\phi_0^4}\left[1+\alpha\left(1-\frac{\phi^2}{\phi_0^2}\right)\right]^{-1}.} \tag{31}$$

We first note that when the rope is fully unwound, that is at $\phi = 0$,

$$\left.\frac{a_B^2}{R^2\Omega^4}\right|_{\phi=0} = (\sqrt{1+\alpha} - 1)^2.$$

Now, since $\alpha = \kappa\phi_0^2/mgh$, let us define the constant γ such that $\gamma = \kappa/mgh$. Therefore, $\alpha = \gamma\phi_0^2$. Thus, if we fix α (such that $0 \leq \alpha \leq 3$) and γ (such that $\gamma > 0$), then the initial twist angle ϕ_0 is pre-determined, where $\phi_0 = \pm\sqrt{\alpha/\gamma}$. A plot for $\frac{a^2}{R^2\Omega^4}$ for several α values where $\gamma = 1$ is shown in Figure 6.

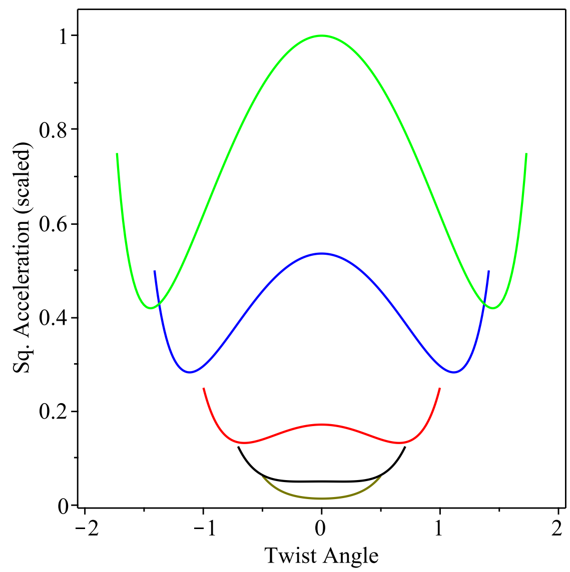

Figure 6: The scaled square acceleration of the bucket wall, $\frac{a_B^2}{R^2\Omega^4}$ (vertical axis), against the twist angle ϕ (horizontal axis) for various α where $\gamma = \kappa/mgh = 1$: $\alpha = 0.25$ (olive), $\alpha = 0.5$ (black), $\alpha = 1$ (red), $\alpha = 2$ (blue), $\alpha = 3$ (green). Since $\gamma = 1$ for all these curves, $\phi_0 = \pm\sqrt{\alpha}$, and therefore, $\phi = [-\phi_0, \phi_0] = [-\sqrt{\alpha}, \sqrt{\alpha}]$.

We observe the following in Figure 6: For a given γ (in this case 1), the twist angle range increases as α increases. This is so since $\phi_0 = \pm\sqrt{\alpha/\gamma}$. For smaller α (e.g., the olive curve where $\alpha = 0.25$) we see a global minimum in a_B^2 at $\phi = 0$ (that is, when the rope is fully unwound). As the α value increases this global minimum in a_B^2 at $\phi = 0$ seems to disappear and become a local maximum (e.g., red curve where $\alpha = 1$); for even higher α (e.g., green curve where $\alpha = 3$) this local maximum at $\phi = 0$ may become a global maximum for a_B^2. We can therefore conjecture that there must be a critical $\alpha = \alpha_c$ where the curves transition through an inflection point at $\phi = 0$. For an inflection point to manifest at $\phi = 0$, the second derivative of (31) with respect to ϕ must vanish there. The associated computation leads to the realization, then, that the condition,

$$16\left(\sqrt{1+\alpha_c} - 1\right)^2 - \frac{\gamma^2}{1+\alpha_c} = 0$$

must be satisfied. Since $\gamma > 0$, the only real solution for α_c that satisfies this condition (using

Maple to evaluate) takes the form,

$$\alpha_c = -\frac{1}{2} + \frac{\sqrt{1+\gamma}}{2} + \frac{\gamma}{4}. \tag{32}$$

Therefore, for a given system determined by $\gamma = \kappa/mgh$, when the rope is fully unwound, the inertial acceleration a_B of the bucket is a global minimum when $\alpha < \alpha_c$; it is a local or a global maximum when $\alpha > \alpha_c$; when $\alpha = \alpha_c$ the inertial acceleration is neither a minimum nor a maximum when the rope has fully unwound. For the curves in Figure 6, since $\gamma = 1$, $\alpha_c \approx 0.46$. Now, we have established that whatever the value of α, it cannot exceed 3, for otherwise the water in the bucket will spill out. Thus, it must be the case that $\alpha_{c_{\max}} = 3$. The corresponding critical $\gamma = \gamma_c$ value is then,

$$\alpha_{c_{\max}} = 3 = -\frac{1}{2} + \frac{\sqrt{1+\gamma_c}}{2} + \frac{\gamma_c}{4}.$$

This condition is only satisfied when,

$$\boxed{\gamma_c = 8.\ \text{Minima to Maxima Non-Transition Condition for Inertial Acceleration}} \tag{33}$$

Therefore, a system with $\gamma_c = \kappa/mgh = 8$ will not demonstrate the inertial acceleration of the bucket transitioning from global minima to local/global maxima unless we allow for water to spill out from it (assuming that the bucket was initially filled with water to its halfway mark). Since for a given γ, $\alpha = \gamma\phi_0^2$, associated with a critical α_c is a critical twist angle ϕ_c. (Note that since $\phi_c^2 = \alpha_c/\gamma$, ϕ_c is only defined for $\gamma > 0$.) The physical significance of the critical twist angle ϕ_c is that if the rope is initially twisted by an amount such that $\phi_0 < \phi_c$, then the inertial acceleration of the bucket reaches a global minimum when the rope is fully unwound. In contrast, if $\phi_0 > \phi_c$, then the inertial acceleration of the bucket reaches a local/global maximum. We summarize these results in the box below:

$$\boxed{\alpha_c = -\frac{1}{2} + \frac{\sqrt{1+\gamma}}{2} + \frac{\gamma}{4}, \quad 0 \le \alpha_c \le 3, \quad 0 < \gamma = \frac{\kappa}{mgh} \le 8, \quad \phi_c = \pm\sqrt{\alpha_c/\gamma}.} \tag{34}$$

The variation of α_c and ϕ_c against γ is shown in Figure 7. When $\gamma = 1$, $\alpha_c = \frac{1}{\sqrt{2}} - \frac{1}{4}$, and therefore, $\phi_c \approx \pm 38.74°$; so, when $\gamma = 1$, if the rope is initially twisted by an angle less than 38.74°, then the inertial acceleration of the bucket will show a global minimum when the rope is full unwound. In contrast, if the rope is initially twisted by an angle greater than 38.74°, then the inertial acceleration of the bucket will show a local or a global maximum when the rope is full unwound. At critical $\gamma = \gamma_c = 8$, $\alpha_c = 3$, and therefore, $\phi_c = \pm\sqrt{3/8} \approx \pm 35.09°$; the bucket's inertial acceleration will therefore have a global minimum when the rope is fully unwound when it is initially twisted by an angle less than 35.09° when $\gamma = 8$. If the rope is twisted above this angle, then there will not be any local or global maxima in inertial acceleration without spilling water out of the bucket (assuming the bucket was initially filled to its halfway mark).

Returning to Figure 6, we finally note that associated with curves having local or global maxima of a_B^2 at $\phi = 0$ are minima that occur symmetrically about $\phi = 0$. These local minima can be found by differentiating (31) with respect to ϕ and picking the stationary points for which the second derivative of (31) with respect to ϕ at those points are positive. However, this exercise is algebraically involved given the higher order polynomials that would result from the differentials. We therefore have to resort to numerical methods to find the local minima. For the curves in Figure 6, these local minima occur at $\phi \approx \pm 0.65$ for $\alpha = 1$, at $\phi \approx \pm 1.12$ for $\alpha = 2$, and at $\phi \approx \pm 1.45$ for $\alpha = 3$.

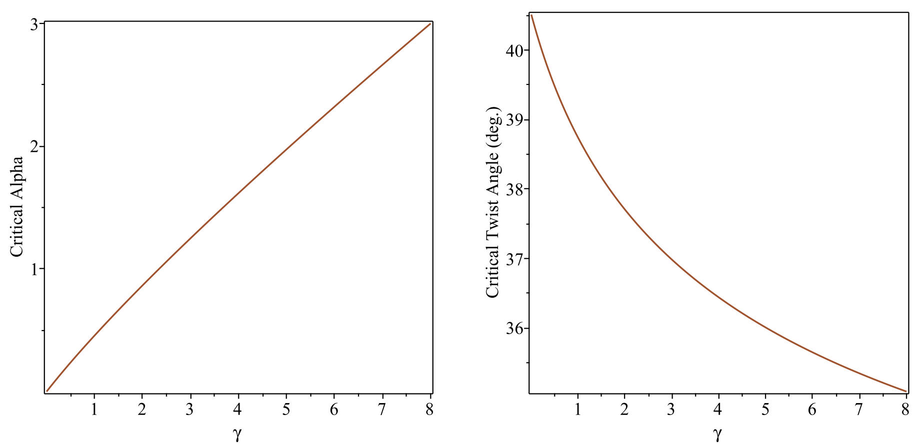

Figure 7: Left: The dependency of α_c as a function of $\gamma = \kappa/mgh$. Right: The dependency of the critical twist angle ϕ_c (in degrees) as a function of $\gamma = \kappa/mgh$. For a given γ, initial twist angles $\phi < \Phi_c$ will lead to global minima in inertial acceleration of the bucket when the rope is fully unwound; similarly, initial twist angles $\phi > \phi_c$ will lead to local/global maxima in inertial acceleration of the bucket when the rope is fully unwound. When $\gamma = \gamma_c = 8$ no local/global maxima can occur without water spilling out of the bucket. Only the positive critical twist angle values are shown; γ values are shown only up to the critical value $\gamma_c = 8$.

2.6 Water Height Dependency on the Twist and the Radial Length

Using (9) and (24) we can express the fractional height of water on the twist angle of the rope and the fractional radial length, which gives

$$\boxed{\frac{z}{h} = 1 - \left\{ \left[1 + \alpha \left(1 - \frac{\phi^2}{\phi_0^2} \right) \right]^{1/2} - 1 \right\} \left(1 - 2\frac{r^2}{R^2} \right).} \tag{35}$$

Note that this expression satisfies the energy condition since we derived the dependency of the angular velocity on the twist angle based on the energy conservation of the system. The fractional water height on the axis (where $r = 0$) then reduces to,

$$\left.\frac{z}{h}\right|_{r=0} = 2 - \left[1 + \alpha \left(1 - \frac{\phi^2}{\phi_0^2} \right) \right]^{1/2}. \tag{36}$$

This function is plotted in Figure 8 for several values of α. We therefore see, that for a given α, the water level on the axis oscillates between $z = h$ (that is, when the rope is twisted to meet the initial condition $\phi = \pm\phi_0$) and a minimum determined by $z = h(2 - \sqrt{1+\alpha})$ (that is, when the rope is fully unwound at $\phi = 0$). We next embark to determine the period of these oscillations.

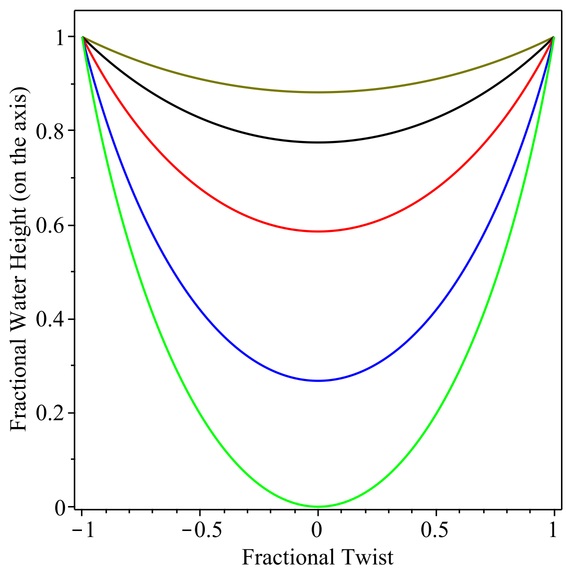

Figure 8: The fractional height of water on the axis of the bucket ($\frac{z}{h}|_{r=0}$; vertical axis) as a function of the fractional twist (ϕ/ϕ_0; horizontal axis) for several values of α: $\alpha = 0.25$ (olive), $\alpha = 0.5$ (black), $\alpha = 1$ (red), $\alpha = 2$ (blue), $\alpha = 3$ (green).

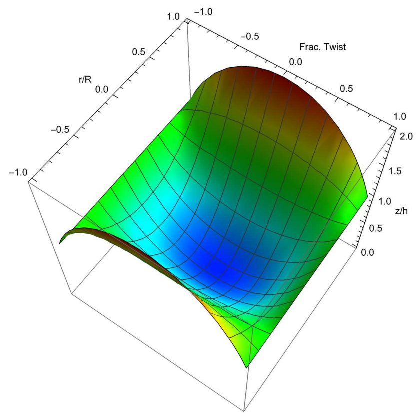

Figure 9: The variation of the fractional height of water (z/h) as a function of fractional twist (ϕ/ϕ_0) and fractional radial length (r/R) when $\alpha = 3$.

2.7 Period of Rotations

As the water bucket twists and untwists it undergoes periodic rotations about the axis. Our goal in this section is to derive an expression for the period of these rotations. For this purpose we start with (27), which is,

$$\omega = \dot{\phi} = \pm\Omega \left\{ \left[1 + \alpha \left(1 - \frac{\phi^2}{\phi_0^2} \right) \right]^{1/2} - 1 \right\}^{1/2} .$$

Let us take $\phi = +\phi_0$ when $t = 0$, which is the amount of twist given to the rope initially in the clockwise direction. If the period of the ensuing rotation of the bucket is T_{ϕ_0}, then at $t = T_{\phi_0}/4$ the rope has to be fully unwound. That is, $\phi = 0$ when $t = T_{\phi_0}/4$. The above expression can then be arranged to read

$$\int_{\phi_0}^{0} \left\{ \left[1 + \alpha \left(1 - \frac{\phi^2}{\phi_0^2} \right) \right]^{1/2} - 1 \right\}^{-1/2} \cdot d\phi = \Omega \int_0^{T_{\phi_0}/4} dt.$$

Now, since $\phi/\phi_0 = [-1, 1]$, we realize that the same range can be achieved by setting $\phi/\phi_0 = \sin\beta$ since $\sin\beta = [-1, 1]$. Therefore, when $\phi = +\phi_0$, $\sin\beta = 1$, and hence, $\beta = \pi/2$; when $\phi = 0$, $\sin\beta = 0$, and hence, $\beta = 0$. Also, since $\phi = \phi_0 \sin\beta$, $d\phi = \phi_0 \cos\beta \cdot d\beta$. Making these substitutions, the above integral becomes,

$$\int_{\pi/2}^{0} \left\{ \frac{\phi_0^2 \cos^2 \beta}{[1 + \alpha \cos^2 \beta]^{1/2} - 1} \right\}^{1/2} \cdot d\beta = \frac{\Omega T_{\phi_0}}{4}.$$

Let $\beta \to -\beta$; therefore, $d\beta \to -d\beta$. Then, since $\cos(-\beta) = \cos\beta$,

$$-\int_{\pi/2}^{0} \left\{ \frac{\phi_0^2 \cos^2(-\beta)}{[1 + \alpha \cos^2(-\beta)]^{1/2} - 1} \right\}^{1/2} \cdot d\beta = \int_{0}^{\pi/2} \left\{ \frac{\phi_0^2 \cos^2 \beta}{[1 + \alpha \cos^2 \beta]^{1/2} - 1} \right\}^{1/2} \cdot d\beta = \frac{\Omega T_{\phi_0}}{4}.$$

We now multiply both sides by $\sqrt{\frac{\kappa}{mgh}}$. Since $\alpha = \frac{\kappa \phi_0^2}{mgh}$, the above yields,

$$\int_{0}^{\pi/2} \left\{ \frac{\alpha \cos^2 \beta}{[1 + \alpha \cos^2 \beta]^{1/2} - 1} \right\}^{1/2} \cdot d\beta = \Omega \cdot \sqrt{\frac{\kappa}{mgh}} \cdot \frac{T_{\phi_0}}{4}.$$

Recall that $I_0 \Omega^2 = 2mgh$. Therefore, the right hand side reduces to $\sqrt{\frac{\kappa}{8I_0}} T_{\phi_0}$. Therefore, the period of the rotations of the bucket is given by,

$$\boxed{T_{\phi_0} = \sqrt{\frac{8I_0}{\kappa}} \int_{0}^{\pi/2} \left\{ \frac{\alpha \cos^2 \beta}{[1 + \alpha \cos^2 \beta]^{1/2} - 1} \right\}^{1/2} \cdot d\beta, \quad 0 \le \alpha \le 3, \quad I_0 = \frac{1}{2} mR^2. \quad \text{Period of Rotations}} \tag{37}$$

The integral on the right hand side does not have a closed form solution and must be evaluated numerically. However, an approximation for the period can be obtained for small initial twists just as for small librations of the simple pendulum. If we expand the square root of the denominator in the integrand using the binomial theorem,

$$\left\{ \frac{\alpha \cos^2 \beta}{[1 + \alpha \cos^2 \beta]^{1/2} - 1} \right\}^{1/2} = \left\{ \frac{\alpha \cos^2 \beta}{\left[1 + \frac{1}{2}\alpha \cos^2 \beta + (\text{higher orders of } \alpha)\right] - 1} \right\}^{1/2}.$$

The higher orders of α will be extremely small given that α is small when ϕ_0 is small. Therefore, the integrand approximates to

$$\left\{ \frac{\alpha \cos^2 \beta}{[1 + \alpha \cos^2 \beta]^{1/2} - 1} \right\}^{1/2} \approx \left\{ \frac{\alpha \cos^2 \beta}{\left[1 + \frac{1}{2}\alpha \cos^2 \beta\right] - 1} \right\}^{1/2} = \sqrt{2}.$$

Therefore, for small ϕ_0

$$T_{\phi_0}\Big|_{\text{small } \phi_0} = \sqrt{\frac{8I_0}{\kappa}} \int_0^{\pi/2} \sqrt{2} \cdot d\beta,$$

which gives,

$$\boxed{T_{\phi_0}\Big|_{\text{small } \phi_0} = T_0 = 2\pi\sqrt{\frac{I_0}{\kappa}} = 2\pi\sqrt{\frac{mR^2}{2\kappa}}. \quad m = \text{water mass}} \tag{38}$$

Just as in the case of small librations of the simple pendulum, we see that for small rotations of the water bucket, T_{ϕ_0} is independent of the initial twist angle ϕ_0. Now, the period for small librations of the simple pendulum is given by $2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum and g is the constant gravitational acceleration. From (38), we see that the period of the small rotations of the water bucket does not depend on g. Therefore, although the period for the small librations of the simple pendulum depends on where it is located, the period for small rotations of the water bucket will be the same anywhere in the universe as long as the moment of inertia of water at rest ($I_0 = \frac{1}{2}mR^2$) and the torsional constant of the rope (κ) remain the same. Figure 10 shows the relative percentage difference in the period as a function of α.

Note that the relative difference in period only depends on α. As the figure shows, the relative percentage difference in the period does not exceed more than about 12.5% even when α is at its maximum value of 3.

2.8 Movement of a Particle on the Water Surface

If a particle is placed on the still surface of the water bucket after the rope has been twisted to the desired degree, then how would this particle move on the water surface as the bucket rotates back and forth? We have deduced earlier that the water surface will start to form a paraboloidal shape as the bucket starts spinning. Further, this shape will change as the angular speed of the bucket varies during a rotational cycle. So, we can expect the particle to move in a complicated trajectory on the surface of the water subject to the initial conditions. One specific motion is intuitively clear, though: that is when the particle is initially placed at the center of the bucket. In this

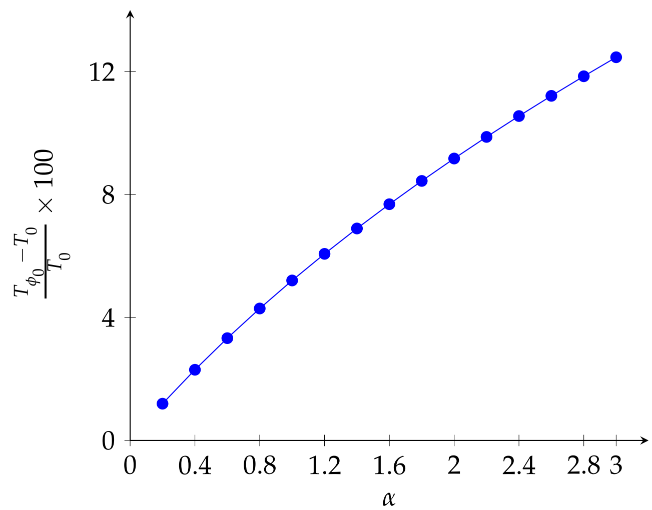

Figure 10: The relative percentage difference in the period as a function of α.

case the particle will oscillate up and down, always staying at the lowest point of the intermediate paraboloids that form as the bucket rotates; its velocity along the axis of the cylinder will be zero when the bucket comes to a rest (that is when the rope has reached its maximum twist, so the water surface is flat) and when the bucket is spinning the fastest (that is when the rope has fully unwound, so the parabolic water surface has reached its maximum depth).

To analyze the motion of the particle let us note from (35) that

$$z = h\left[1 - \left\{\left[1 + \alpha\left(1 - \frac{\phi^2}{\phi_0^2}\right)\right]^{1/2} - 1\right\}\left(1 - 2\frac{r^2}{R^2}\right)\right]. \tag{39}$$

This expression takes into account the variation of the z coordinate of the particle (on the water surface) subject to its radial distance $r(t)$ from the axis and the azimuthal angle $\phi(t)$ the bucket makes at any given instant of time $t = t$. Since this provides a constraint for the motion of the particle such that it is confined to move only on the water surface, the particle has only two degrees of freedom. Let us now differentiate the above with respect to time to obtain the velocity of the particle along the axis. The resulting expression is:

$$\dot{z} = h\left[\frac{\alpha\,\phi\dot{\phi}}{{\phi_0}^2}\left(1 - 2\,\frac{r^2}{R^2}\right)\frac{1}{\sqrt{1 + \alpha\left(1 - \frac{\phi^2}{{\phi_0}^2}\right)}} + 4\,\frac{r\dot{r}}{R^2}\left\{\sqrt{1 + \alpha\left(1 - \frac{\phi^2}{{\phi_0}^2}\right)} - 1\right\}\right] \tag{40}$$

Since $x = r\cos\phi$ and $y = r\sin\phi$, the particle has velocities

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi, \tag{41}$$
$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \tag{42}$$

in the x and the y directions, respectively. The square of the speed of the particle is then $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$. The kinetic energy of the particle is $\mathcal{E}_k = \frac{1}{2}m_\bullet v^2$ where $m_\bullet$ is the mass of the particle. The potential energy of the particle is $\mathcal{E}_p = m_\bullet g z$ where the zero potential energy level has been taken as the bottom of the bucket which is on the $X - Y$ plane. Subtracting $\mathcal{E}_p$ from $\mathcal{E}_k$ and multiplying the resulting equation by $2/m_\bullet$, the Lagrangian for the particle on the water surface is

$$\mathcal{L} = v^2 - 2gz = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2gz,$$

where the variables on the right are given by the expressions (39) – (42). Hence, the full form of the Lagrangian reads

$$\mathcal{L} = \dot{r}^2 + r^2\dot{\phi}^2 + h^2\left[\frac{\alpha\,\phi\dot{\phi}}{{\phi_0}^2}\left(1 - 2\,\frac{r^2}{R^2}\right)\frac{1}{\sqrt{1+\alpha\left(1 - \frac{\phi^2}{{\phi_0}^2}\right)}} + 4\,\frac{r\dot{r}}{R^2}\left\{\sqrt{1+\alpha\left(1 - \frac{\phi^2}{{\phi_0}^2}\right)} - 1\right\}\right]^2$$
$$+ 2gh\left[1 - \left\{\sqrt{1+\alpha\left(1 - \frac{\phi^2}{\phi_0^2}\right)} - 1\right\}\left(1 - 2\frac{r^2}{R^2}\right)\right]. \tag{43}$$

Thus we see that the Lagrangian of the particle has only r and ϕ as the generalized coordinates reflecting the fact that it has only two degrees of freedom. The corresponding equations of motion can be found via the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} = 0, \quad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

In this analysis we have used Maple to obtain the equations of motion, which we will not display due to their length as the reader may intuit given the form of the Lagrangian. The equations of motion can then be solved numerically using the Runge-Kutta method. Figure 11 shows the movement of the particle on the water surface for a particular set of initial conditions.

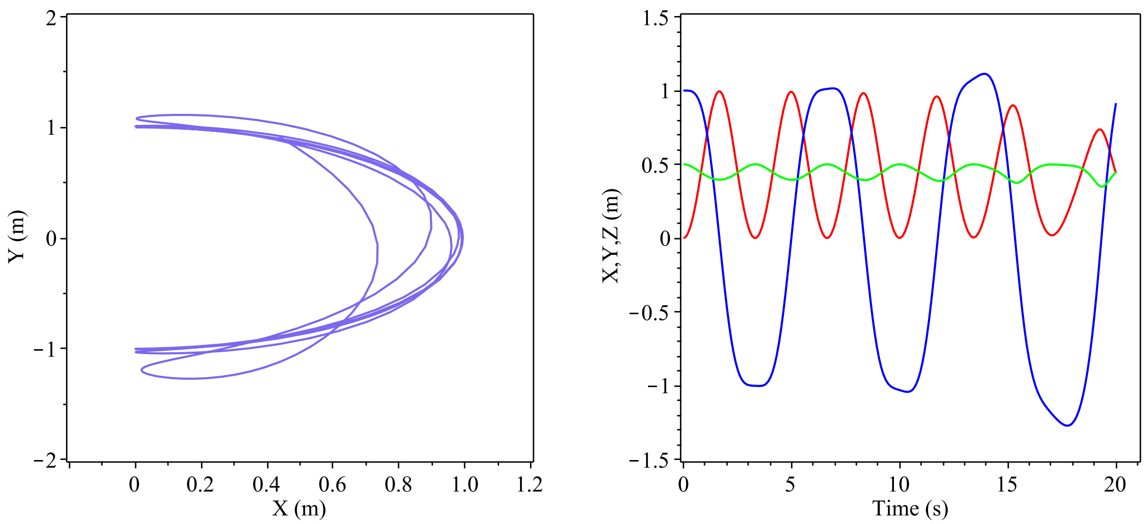

Figure 11: The movement of a particle on the water surface of the spinning bucket with initial conditions: $\alpha = 1$, $R = 2$ (radius of the bucket), $h = 1/2$ (still water height), $r(0) = 1$ (initial radial distance from the axis), $\phi_0 = \pi/2$ rad (initial twist angle of the rope), $\dot{r}(0) = 0$ (initial radial velocity), $\dot{\phi}(0) = 0$ (initial angular velocity), where distances are in meters and $g = 9.81$ ms^{-2}. Thus the rope has been initially twisted by 90°; this means that if a point on the bottom rim of the bucket is on the x-axis when the rope is unwound, then that point will be on the y-axis when the rope has been twisted. The particle has been placed half the radius of the bucket away from the axis such that the vertical line through the particle parallel to the axis of the bucket meets the y-axis (due to the initial 90° twist). Left: The trajectory of the particle projected to the X-Y plane. Right: The variation in the x (X, red), y (Y, blue), and z (Z, green) coordinates against time. The green curve captures the bobbing of the particle on the water surface as the depth of the water surface varies during a rotational cycle. All trajectories have been drawn for a duration of 20 s.

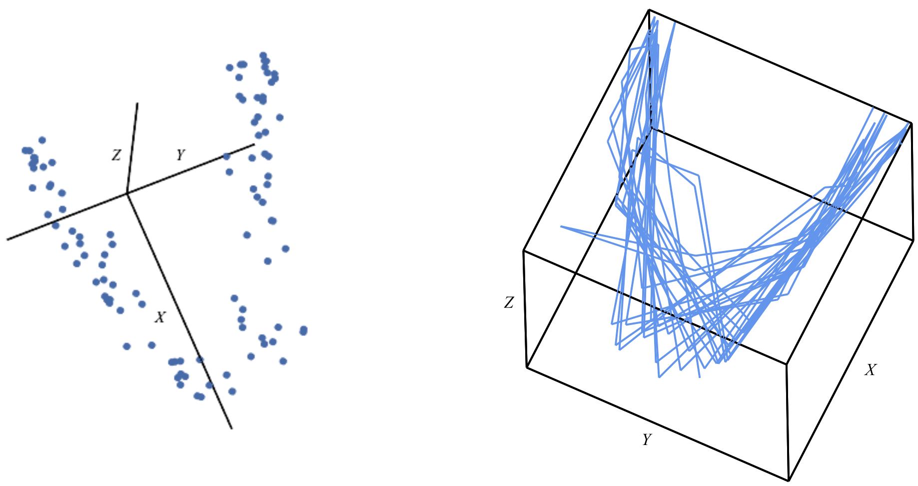

Figure 12: The movement of the particle on the water surface of the spinning bucket with the same set up as described in Figure 11. Left: Three-dimensional snapshots of the particle taken for a duration of 10 s of its motion at intervals of 1/10 s. Right: The connected set of points of the figure on the left.

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