# Where Would It Fall?
# The Problem of Motion

Rasil Warnakulasooriya

## Contents



## 1 A Snippet from History

Ever since humans gazed up they have grappled to come to grips with the motion of the objects in the sky. The Sun, the moon, the planets, and the stars all move in a seemingly regular motion around the Earth. The apparent motion of these bodies around the Earth is largely responsible for the belief of the ancients that the Earth is at the center of the universe; hence the geocentric universe. Similarly, the difficulty of feeling or observing that the Earth is moving through space gave rise to the belief that the Earth is immovable; hence the geostatic universe. We invite the reader to transform back in time, and imagine, without all the current understanding about the universe, the celestial and terrestrial movements of objects – what conclusions would you reach regarding these objects and their motions, including that of the Earth?

Earthlings do not experience motion of the Earth about its axis or around the Sun, and hence, that it is not in motion was a belief-system that was natural. Claudius Ptolemy (c. 100 – c. 170 AD) positioned the Earth at the center of the universe and at rest in his text, *Almagest*. In Ptolemy's geocentric and geostatic model of the universe the Sun, the moon, and the planets moved around the Earth in complicated orbits needed to account for the observations such as the retrograde[1] motion of Mars, his ultimate goal being the quantitative prediction of the positions of the heavenly bodies at any given time as observed from the Earth.

Long before Nicholas Copernicus (1473 – 1543) moved the Sun to the center of the universe and made the Earth move around it, Ptolemy was aware that one can make the Earth spin about its

[1] Retrograde motions are the apparent reversals in motion of the planets as observed from the Earth.

axis to account for the daily motions of the objects in the sky. Here, however, Ptolemy ran into a great difficulty regarding the problem of motion of the Earth. If the Earth spun about its axis, "animals and other weights would be left hanging in the air, and the Earth would very quickly fall out of the heavens. Merely to conceive such things makes them appear ridiculous."[2] This statement by Ptolemy shows one of the great challenges in detecting the motion of an object while being a rider of that object, in this case, the Earth. Thus, making the Earth go around the Sun according to the Copernican heliocentric model does not trivially resolve the objections to its motion. Critiques argued that if the Earth is in motion, then an object thrown straight up should[3] land behind the thrower, for during the time the object is in flight the Earth has moved past the point in space at which it was thrown. Our common experience shows that this is not the case – a ball thrown straight up will land in the hands of the thrower. Tea poured inside an airplane, which moves at a constant speed smoothly on a straight path will land in a cup just as it would land in the comfort of one's home.

With all we know today about Earth's motion around the Sun, it is easy for us to not appreciate the difficulty faced by the early physicists[4] regarding its motion. The apparent inability to detect Earth's motion in reference to other objects in the sky was extended by Galileo Galilei (1564 – 1642) in his text *Dialogue Concerning the Two Chief World Systems*. In the *Dialogue*, a conversation and a debate take place among three individuals: Salviati, who argues for the heliocentric model of Copernicus; Simplicio, who argues for the geocentric model of Ptolemy and other ancient ideas of motion; and Sagredo, a layman, who listens to both sides of the argument posing intelligent questions to both Salviati and Simplicio. In it, Salviati asks Simplicio and Sagredo to imagine a ship moving on the calm seas[5]:

> For a final indication of the nullity of the experiments brought forth, this seems to me the place to show you a way to test them all very easily. [The nullity that Salviati alludes to is the difficulty of distinguishing a state of motion from a state of rest.] Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a narrow-mouthed vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though there is no doubt that when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you

[2] *The Eye of Heaven*, Owen Gingerich, American Institute of Physics, New York, 1993, pg. 5.

[3] We take that the effects of air resistance can be ignored for all practical purposes. Ideally, all experiments of concern take place in a vacuum, without worrying how humans and other beings can function in such an environment.

[4] The field of physics and the practitioners of it, physicists, were called natural philosophy and natural philosophers, respectively, in the past. The modern terms became established toward the late 19th century.

[5] *Dialogue Concerning the Two Chief World Systems*, Galileo Galilei; Stillman Drake (translator), Stephen J. Gould (series editor), The Modern Library, New York, 2001, pg. 216-218.

> tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other. The cause of all these correspondences of effects is the fact that the ship's motion is common to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this tool place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted. No doubt the smoke would fall as much behind as the air itself. The flies likewise, and the butterflies, held back by the air, would be unable to follow the ship's motion if they were separated from it by a perceptible distance. But keeping themselves near it, they would follow it without effort or hindrance; for the ship, being an unbroken structure, carries with it a part of the nearby air. For a similar reason we sometimes, when riding horseback, see persistent flies and horseflies following our horses, flying now to one part of their bodies and now to another. But the difference would be small as regards the falling drops, and as to the jumping and the throwing it would be quite imperceptible.

After four days of conversation and debate where varying degrees of understanding have been reached, and more questions remain unanswered, Sagredo declares: "... according to our custom, let us go and enjoy an hour of refreshment in the gondola that awaits us." Thus ends Galileo's *Dialogue*, which argues in favor of Copernicus' heliocentric model and highlights the inability to distinguish physical phenomena according to observers' state of motion. By implication, to Galileo, this means that the Earth can very well move around the Sun although that motion may not be detectable for those living on it. The reader may detect some challenges here. What is indistinguishable is *uniform motion* (that is, motion at a constant speed along a straight line) from that of the state of rest. An object moving along a circle does not qualify as uniform motion due to the changing nature of the direction of travel. However, Earth's motion about the Sun can be taken as approximately uniform, as we will show later. Galileo's argument, then, is that not feeling the motion of the Earth is not a good reason to reject its motion around the Sun. If we are to consider only the indistinguishability of uniform motion from that of rest, then we can summarize Galileo's argument as a principle of nature:

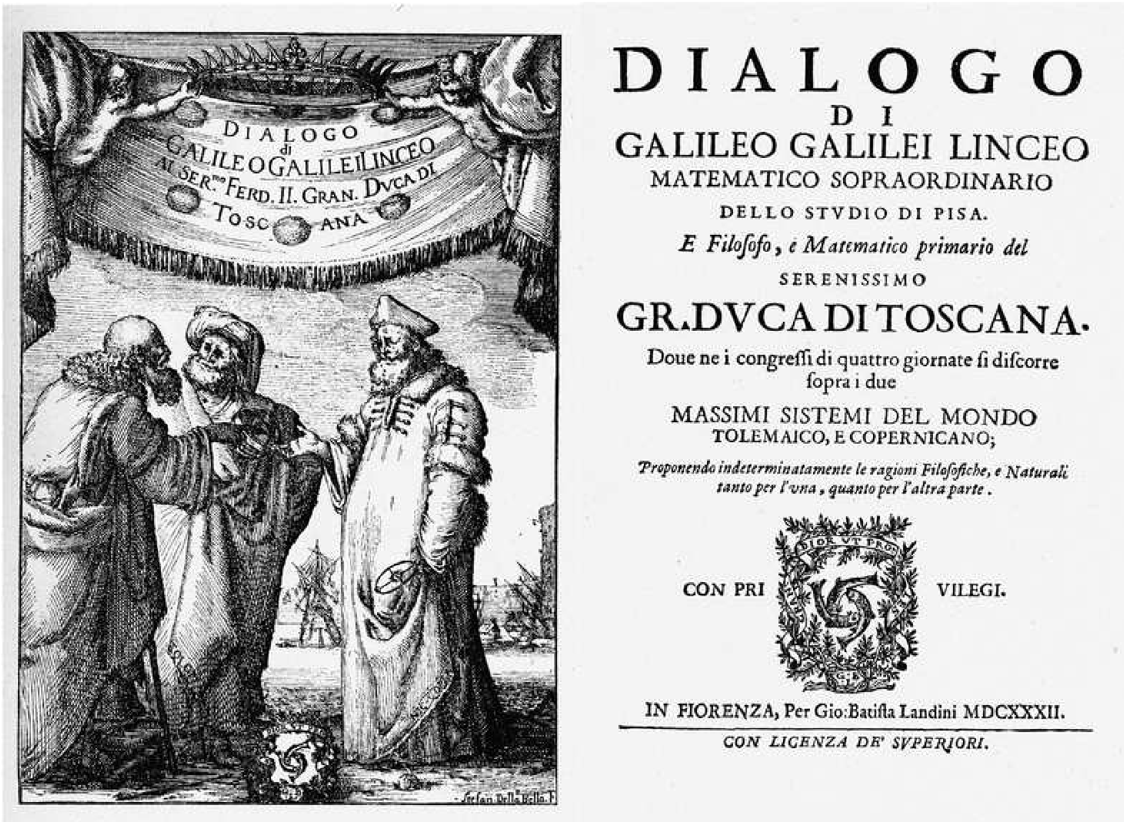


Figure 1: The frontispiece and the title page of Galileo Galilei's *Dialogue Concerning the Two Chief World Systems* (1632) shows Aristotle (left), Ptolemy (middle), and Copernicus (right) in debate and discussion. Ptolemy holds a model of his geocentric model of the universe while Copernicus holds his heliocentric model.

**Galileo's Principle of Relativity**

Mechanical laws of nature are the same for observers in uniform motion and for observers at rest. Stated differently, there is no experiment one can perform which will indicate whether one is in uniform motion or is at rest.

The term *Mechanical laws* is important in Galileo's principle of relativity since the understanding of the electromagnetic phenomena were at its infancy during his time and holds the key to its extension by the later protagonists in the story of relativity. As expounded by Galileo, it is not possible to distinguish uniform motion from that of rest using any mechanical experiments. This is a profound step in the understanding of nature, especially when set in its historical context.

At the time of Galileo, the prevailing notions of how the world works were based on the physics of Aristotle (384 BC – 322 BC), which in turn were based on commonsense experiences combined with logic. To Aristotle, the state of rest is natural, whereas motion is unnatural. Thus, to Aristotle, there is a fundamental distinction between being at rest, and being in motion. Galileo's principle of relativity removes this distinction putting both the state of rest and the state of uniform motion on an equal footing.

Putting the state of rest and the state of uniform motion on an equal footing, however, raises the question of *with respect to what do we observe a state of rest or a state of uniform motion?* The answer to this question taxed Isaac Newton (1643 – 1727), who proposed that the states of motion (rest, uniform motion, and acceleration) are relative to the immovable space, which he called absolute space, within which all things in the universe are embedded in; therefore, rest or motion can have an absolute meaning with regards to the question of "with respect to what?" However, since one cannot do any mechanical experiment to distinguish whether one is at rest or is moving in uniform motion with respect to absolute space, this raises an epistemological – that is, "how do we know what we know" – question according to Gottfried Wilhelm Leibniz (1646 – 1716). Since the existence of absolute space cannot be established given the indistinguishability of the states of rest and uniform motion, Leibniz's proposal was to disregard absolute space and consider only the relative motions between observers. That is, only the motions that one can observe with respect to one another matters. A passenger on a uniformly moving train is free to consider him or herself as at rest and the station to be moving at a uniform speed along a straight line relative to the train. A person standing still relative to the station is free to consider him or herself as at rest and the train to be moving at a uniform speed along a straight line relative to the station. Both will obtain the same results if they perform identical mechanical experiments: for example, tea poured into a cup will land in the cup without spilling on the floor, a ball thrown straight up will land straight back in the hands, and a cord hanging straight down from the respective ceilings will remain vertical without slanting to one side or another.

Galileo's principle of relativity very nearly applies to mechanical phenomena on Earth. Therefore, the Earth can be effectively considered to be in uniform motion so that it is not trivial to establish its movement through space using earthly observations. As described earlier, the thinking of the ancients that the Earth is immovable arose from observing daily phenomena, one of which is the falling of bodies. If one throws a rock vertically upwards, then one is able to catch it when it comes back down. How would that be possible if the Earth is moving while the rock is up in the air? Wouldn't the rock land "behind" the thrower if the Earth is moving?

The equatorial radius of the Earth is about $6,378$ km ($\approx 3,963$ miles). Considering Earth's diurnal (daily) rotation, it takes about 24 hours for a point on Earth's surface to complete a full revolution about its axis. Therefore, a person at the equator rotates at a speed of $6378 \times 2\pi/24 \approx 1670$ km/hr ($\approx$ 1038 miles/hr). This is equivalent to about 1522 feet/s. Now, if this person at the equator throws a rock vertically, which, say, takes one second to go up, then it will take another second to come back down. But during this two-second time interval, the Earth, and hence our thrower, has moved about 3044 feet ($\approx$ 0.58 miles or 0.93 km) from the point of launch. So, wouldn't the rock land close to a kilometer or half-a-mile "behind" the thrower? If one takes into account the motion of the Earth around the Sun, the effect is even more dramatic. As one may verify with an orbital radius of about $150 \times 10^6$ km ($93 \times 10^6$ miles), the Earth moves at a speed of about 30 km/s ($\approx$ 19 miles/s) along its nearly circular orbit. This implies, per the earlier experiment, that the thrown rock has to land about 60 km ($\approx$ 38 miles) "behind" the thrower. Our experience, however, contradicts these expectations; rocks thrown straight up seem to land straight down in a such a way that we are able to catch them; games that involve throwing or hitting balls would otherwise not be feasible.

Since Earth's angular speed due to its rotation is quite steady, its angular acceleration nearly vanishes; thus, the acceleration tangential to the Earth's surface due to its rotation nearly vanishes. Also, since the angular speed of the Earth's rotation about its axis is small [$2\pi$ rad/(86400 s) $\approx 7.3 \times 10^{-5}$ rad/s] its square is even smaller; hence, the radial acceleration of a point on the Earth's surface is also small; it is about $6378 \times 10^3 \times (7.3 \times 10^{-5})^2 \approx 0.03$ m/s$^2$. Therefore, the acceleration of the Earth due to its daily rotation is practically negligible. Considering the motion of the Earth around the Sun in its nearly circular orbit, again, the angular speed due to its orbital motion is quite steady; thus the acceleration tangential to the Earth's orbit nearly vanishes. What is left is largely the centripetal acceleration toward the Sun, which is also small; it is about $150 \times 10^9 \times (2 \times 10^{-7})^2 \approx 0.006$ m/s$^2$. Therefore, the acceleration of the Earth due to its orbital motion is also practically negligible. So, for the dwellers on Earth, it can be effectively considered as a uniformly moving platform in space. In other words, short-duration projectiles would not easily reveal the motion of the Earth itself, just as Galileo suspected.

Our goal in this essay is to analyze the motion of falling bodies as described by observers in platforms that are either at rest or are moving. Regarding movement, we will consider both uniform and accelerated motion. For this purpose we must first introduce reference frames relative to which all motion is analyzed. Among these reference frames, a special class known as inertial frames play a fundamental role in that the Galilean principle of relativity and Newton's second law of motion are valid only in such frames of reference. We take inertial frames to be either at rest or to be moving uniformly in a straight line with respect to absolute space. Therefore, if two frames are inertial with respect to absolute space, then they are inertial with respect to each other. We will then show how Newton's second law is modified in non-inertial frames. These non-inertial frames may be translating and/or rotating with respect to inertial frames. Four types of motion are analyzed using the resulting expressions: (a) a ball thrown vertically upward in a train car moving at constant velocity relative to the station, (b) a ball thrown vertically upward in a train car moving at constant acceleration relative to the station, (c) a ball dropped from a tower, and (d) a ball thrown vertically upward at a point on the surface of the Earth.

## 2 Reference Frames: Inertial & Non-Inertial

All motion must be described with respect to reference frames. A reference frame is defined as consisting of an origin and three orthogonal unit vectors[6] whose directions are fixed in the frame. Therefore, any time derivative of the unit vectors of a frame taken with respect to the frame itself vanishes. Let us formalize this important definition as follows:

**Reference Frame: definition**

In three-dimensional space, let $\mathcal{R}$ be a reference frame defined by the origin $O$ and the fixed orthogonal unit vector triad $\mathbf{e}_i$ ($i = 1, 2, 3$). Thus, $\mathcal{R} \equiv \{O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where distances to points in space are measured with respect to $O$ and

$$\frac{{}^{\mathcal{R}}d}{dt}\mathbf{e}_i = \mathbf{0}, \quad i = 1, 2, 3.$$

Here, the operator $\frac{{}^{\mathcal{R}}d}{dt}$ denotes the fact that the time derivative is taken with respect to the reference frame $\mathcal{R}$.

It is important to note that the unit vectors $\mathbf{e}_i$ in the above definition are only fixed in the associated reference frame $\mathcal{R}$. These unit vectors may be changing over time with respect to a different reference frame $\mathcal{R}'$, in which case $\frac{{}^{\mathcal{R}'}d}{dt}\mathbf{e}_i \neq \mathbf{0}$. We now define the inertial reference frame:

**Inertial Reference Frame: definition**

An inertial reference frame, $\mathcal{I}$, is a frame of reference that is either at rest or moves at a constant velocity with respect to absolute space; therefore, its defining orthogonal unit vector triad is fixed in relation to absolute space (practically, in relation to fixed stars). Reference frames that are inertial with respect to absolute space are inertial to each other.

The Galilean principle of relativity and the form $\mathbf{F} = m\mathbf{a}$ of Newton's second law apply only to mechanical phenomena observed in inertial frames of reference. Here, $\mathbf{F}$ is the total force on a point particle, $m$ is the mass of the particle, and $\mathbf{a}$ is the acceleration of the particle as measured with respect to an inertial frame of reference.

In Newtonian mechanics, once synchronized, the clocks in two reference frames read the same time irrespective of their state of motion. This is the notion that Newton stated as absolute time that is common to all observers in reference frames. In other words, in Newtonian mechanics, time is not relative. Absolute time (as well as absolute space) is removed in Einstein's theory of relativity where (once synchronized) clock readings between events differ in reference frames with differing states of motion (which is the case even among different inertial frames). Therefore,

[6] We denote vectors in bold font; hence, $\mathbf{v}$ is a vector with magnitude $v$.

in Einstein's relativity, time is relative.

# 3 Newton's Second Law: Translations

Let us denote the position vector of a point $P_i$ with respect to another point $P_j$ in absolute space as $\mathbf{r}_{P_i/P_j}$. Hence, $\mathbf{r}_{P_i/P_j}$ can be represented as a line with an arrowhead at the end which extends from $P_j$ to $P_i$. The starting point of all computations in kinematics and dynamics, in the Newtonian framework, is the identification of position vectors of particles with respect to the origins of frames of reference. The time derivatives of these position vectors with respect to various frames would then yield the necessary velocities and accelerations of the particles as measured in those frames.

We first put this approach to practice in the following context where we derive the form of Newton's second law in a reference frame which moves rectilinearly with respect to an inertial frame. Such a frame is said to be in translation with respect to the inertial frame. Further, let us take the motion of a particle to take place on the (two-dimensional) plane with the inertial and translating frames also located on the same plane such that the orthogonal unit vector triads of the inertial and the translating frames are parallel to each other. Our goal, then, is to describe the motion of the particle as observed in these two frames. Figure 2 shows the set up in detail.

First, let us describe the motion of the point particle $P$ with respect to the inertial frame $\mathcal{I} \equiv \{O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. Note that we are considering a right-handed coordinate system so that $\mathbf{e}_z$ points out of the X-Y plane (page). Let the position vector of the particle with respect to the origin of the inertial frame be $\mathbf{r}_{P/O}$. If the coordinates of the particle as observed in the inertial frame is $P = (x, y, 0)_{\mathcal{I}}$ ($z = 0$ since the motion takes place on the X-Y plane), then

$$\mathbf{r}_{P/O} = x\mathbf{e}_x + y\mathbf{e}_y + 0\mathbf{e}_z = x\mathbf{e}_x + y\mathbf{e}_y. \tag{1}$$

Let us now compute the velocity of the particle at any instant with respect to the inertial frame. This is given by,

$$\frac{^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{P/O} = \left(\frac{^{\mathcal{I}}d}{dt}x\right)\mathbf{e}_x + x\left(\frac{^{\mathcal{I}}d}{dt}\mathbf{e}_x\right) + \left(\frac{^{\mathcal{I}}d}{dt}y\right)\mathbf{e}_y + y\left(\frac{^{\mathcal{I}}d}{dt}\mathbf{e}_y\right).$$

Now,

$$\frac{^{\mathcal{I}}d}{dt}x = \frac{dx}{dt} = \dot{x}, \quad \frac{^{\mathcal{I}}d}{dt}\mathbf{e}_x = \mathbf{0}; \quad \frac{^{\mathcal{I}}d}{dt}y = \frac{dy}{dt} = \dot{y}, \quad \frac{^{\mathcal{I}}d}{dt}\mathbf{e}_y = \mathbf{0}.$$

Since $x$ and $y$ are scalars, it does not matter with respect to what frame of reference the time derivatives are taken. Therefore $\dot{x}$ and $\dot{y}$ do not have the frame index on them. The time derivative

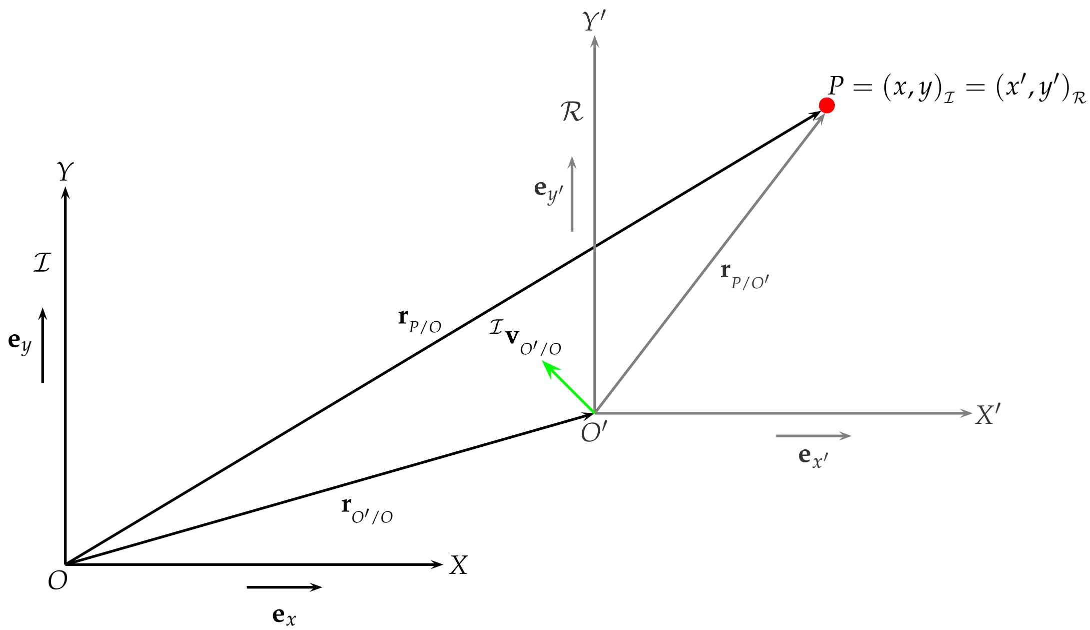


Figure 2: A reference frame $\mathcal{R} \equiv \{O', \mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}\}$ translating (hence, not rotating) on the plane relative to the inertial reference frame $\mathcal{I} \equiv \{O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. The green arrow shows the instantaneous velocity (${}^{\mathcal{I}}\mathbf{v}_{O'/O}$) of the translating reference frame relative to the inertial frame. The goal is to describe the motion of the point particle $P$ as observed in the two frames of reference; the coordinates of $P$ as observed in the respective reference frames are: $P = (x, y)_{\mathcal{I}} = (x', y')_{\mathcal{R}}$. The two reference frames and the particle are situated on the same plane. The unit vector triads of the two frames are oriented such that $\mathbf{e}_x = \mathbf{e}_{x'}$, $\mathbf{e}_y = \mathbf{e}_{y'}$, and $\mathbf{e}_z = \mathbf{e}_{z'}$. The unit vectors $\mathbf{e}_z$ and $\mathbf{e}_{z'}$ point out of the plane (page). The respective position vectors are marked. | Drawing by AD.

of unit vectors vanish since the derivatives are taken with respect to the inertial frame in which the unit vectors are fixed. Thus, the velocity of the particle with respect to the inertial frame is

$$\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{P/O} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y \tag{2}$$

Taking the time derivative of the above expression with respect to the inertial frame and following the same approach, the acceleration of the particle at any instant with respect to the inertial frame is

$$\frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{a}_{P/O} = \ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y. \tag{3}$$

Now, in an inertial frame, Newton's second law applies by definition:

**Newton's Second Law in an Inertial Reference Frame: definition**

If $P$ is a point particle with mass $m_P$, $\mathbf{F}_P$ is the total force acting on it, and ${}^{\mathcal{I}}\mathbf{a}_{P/O}$ is the acceleration of $P$ as determined by an observer located at the origin $O$ of the inertial frame $\mathcal{I}$, then Newton's second law in the inertial reference frame takes the form

$$\mathbf{F}_P = m_P\,{}^{\mathcal{I}}\mathbf{a}_{P/O}\,. \tag{4}$$

This, of course, is the familiar $\mathbf{F} = m\mathbf{a}$, which is based on experimental evidence and cannot be derived. Aristotle believed that force is proportional to speed, and therefore, in the physics of Aristotle, an ever-present (net) force is required for an object to be in motion. It took over a thousand years from the time of Aristotle to arrive at the realization that the true nature of motion is such that force is proportional to acceleration, and therefore, an object will move rectilinearly at constant speed even in the absence of any net force on the object.

Note that Newton's second law applies only to point particles where we do not consider any dimensions of the bodies concerned. However, this is not an impractical abstraction since many questions in mechanics can be solved by modeling large objects as point particles; for example, given the vast distance between them, the Earth can be considered as a point particle when analyzing its motion around the Sun.[7] The reader may wonder at this point about the seemingly cumbersome notation that we have adopted. This notation, however, will pay great dividends as we analyze more complicated motions since it allows us to keep track of what we are doing and with respect to which frames of reference every step of the way. This ultimately allows us to be systematic in our approach to kinematics and dynamics, and thereby forego somewhat ad hoc methods of reasoning typically utilized in analyzing motion.

[7] The Earth's radius is only about 0.004% compared to its orbital radius.

Now that we have defined Newton's second law for a point particle with respect to the inertial frame, let us look at its form with respect to the translating frame $\mathcal{R} \equiv \{O', \mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}\}$. For this, we need to consider the position vector of the particle with respect to the origin of the translating frame. Applying vector addition to the triangle formed by the points $OPO'$ (see Figure 2), we have

$$\mathbf{r}_{P/O} = \mathbf{r}_{O'/O} + \mathbf{r}_{P/O'} \ .$$

Taking the time derivative of this expression with respect to the inertial frame yields,

$$\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} = \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{O'/O} + \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O'} \ .$$

Now,

$$\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{P/O} \ \text{ and } \ \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{O'/O} = {}^{\mathcal{I}}\mathbf{v}_{O'/O} \ ,$$

where ${}^{\mathcal{I}}\mathbf{v}_{P/O}$ is the velocity of the particle with respect to the inertial frame $\mathcal{I}$ and ${}^{\mathcal{I}}\mathbf{v}_{O'/O}$ is the velocity of the translating frame with respect to the inertial frame $\mathcal{I}$. To find $\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O'}$ , note that since $P = (x', y')_{\mathcal{R}}$ ,

$$\mathbf{r}_{P/O'} = x'\mathbf{e}_{x'} + y'\mathbf{e}_{y'}.$$

Now, recall that the unit vector triads between the two frames were oriented such that they are parallel. Therefore, $\mathbf{e}_{x'} = \mathbf{e}_x$, $\mathbf{e}_{y'} = \mathbf{e}_y$, $\mathbf{e}_{z'} = \mathbf{e}_z$, and remains so since frame $\mathcal{R}$ is only translating (and hence, not rotating) on the plane with respect to the inertial frame $\mathcal{I}$. Thus

$$\mathbf{r}_{P/O'} = x'\mathbf{e}_x + y'\mathbf{e}_y.$$

Since the time derivative with respect to the inertial frame $\mathcal{I}$ does not affect the unit vectors attached to it,

$${}^{\mathcal{I}}\mathbf{v}_{P/O'} = \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O'} = \dot{x}'\mathbf{e}_x + \dot{y}'\mathbf{e}_y = \dot{x}'\mathbf{e}_{x'} + \dot{y}'\mathbf{e}_{y'} = \frac{{}^{\mathcal{R}}d}{dt}\mathbf{r}_{P/O'} = {}^{\mathcal{R}}\mathbf{v}_{P/O'} \ . \tag{5}$$

Combining these results we obtain the Newtonian velocity addition formula ${}^{\mathcal{I}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{O'/O} + {}^{\mathcal{R}}\mathbf{v}_{P/O'}$ . We can similarly reason that ${}^{\mathcal{I}}\mathbf{a}_{P/O'} = {}^{\mathcal{R}}\mathbf{a}_{P/O'}$ , which leads to the Newtonian acceleration addition formula ${}^{\mathcal{I}}\mathbf{a}_{P/O} = {}^{\mathcal{I}}\mathbf{a}_{O'/O} + {}^{\mathcal{R}}\mathbf{a}_{P/O'}$ . We now note that the velocity and the acceleration addition formulae are vector equations. Since vectors are geometric objects, vector equations are valid beyond the planar context we have utilized to derive them. For example, the content of the velocity addition formula ${}^{\mathcal{I}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{O'/O} + {}^{\mathcal{R}}\mathbf{v}_{P/O'}$ is that if the velocity (${}^{\mathcal{R}}\mathbf{v}_{P/O'}$) of a particle is observed in a reference frame $\mathcal{R}$, and if $\mathcal{R}$ is translating in space relative to an inertial frame $\mathcal{I}$ at a certain velocity (${}^{\mathcal{I}}\mathbf{v}_{O'/O}$), then the velocity (${}^{\mathcal{I}}\mathbf{v}_{P/O}$) of the particle as observed in $\mathcal{I}$ is the vector sum of ${}^{\mathcal{R}}\mathbf{v}_{P/O'}$ and ${}^{\mathcal{I}}\mathbf{v}_{O'/O}$. As such, this vector sum holds even when $O$, $O'$, and $P$ are not on the same plane. Furthermore, the relative orientation between the two frames can be determined by finding the correspondence among the respective unit vectors; we will demonstrate how to do this later in the essay in the context of observing the fall of a ball from a lab frame on Earth. We summarize these important results below.

**Newtonian Velocity & Acceleration Addition for Translations**

If $\mathcal{I}$ is an inertial reference frame with origin $O$ and $\mathcal{R}$ is a reference frame with origin $O'$ that is translating relative to $\mathcal{I}$, and $P$ is a point particle, then

$$ {}^{\mathcal{I}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{O'/O} + {}^{\mathcal{R}}\mathbf{v}_{P/O'} \,, \tag{6} $$

$$ {}^{\mathcal{I}}\mathbf{a}_{P/O} = {}^{\mathcal{I}}\mathbf{a}_{O'/O} + {}^{\mathcal{R}}\mathbf{a}_{P/O'} \,. \tag{7} $$

We can now multiply (7) by the mass of the particle to obtain

$$ m_P \, {}^{\mathcal{I}}\mathbf{a}_{P/O} = m_P \, {}^{\mathcal{I}}\mathbf{a}_{O'/O} + m_P \, {}^{\mathcal{R}}\mathbf{a}_{P/O'} \,, $$

where we note from (4) that $m_P \, {}^{\mathcal{I}}\mathbf{a}_{P/O} = \mathbf{F}_P$. Thus,

**Newton's Second Law in a Translating (Non-Rotating) Reference Frame**

If $\mathcal{I}$ is an inertial reference frame with origin $O$ and $\mathcal{R}$ is a reference frame with origin $O'$ that is translating relative to $\mathcal{I}$, and $P$ is a point particle, then Newton's second law in the translating frame $\mathcal{R}$ takes the form

$$ m_P \, {}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{F}_P - m_P \, {}^{\mathcal{I}}\mathbf{a}_{O'/O} \,. \tag{8} $$

Note that if the translating reference frame $\mathcal{R}$ is accelerating relative to the inertial frame $\mathcal{I}$, that is, if ${}^{\mathcal{I}}\mathbf{a}_{O'/O} \neq \mathbf{0}$, then (8) differs from the form in (4). In contrast, if it is translating at a constant velocity relative to the inertial frame, in which case ${}^{\mathcal{I}}\mathbf{a}_{O'/O} = \mathbf{0}$, then the forms (8) and (4) agree and Newton's second law in this case reads $m_P \, {}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{F}_P$. Since this form only applies in an inertial frame of reference, this means that $\mathcal{R}$ is an inertial reference frame when it is translating at constant velocity relative to $\mathcal{I}$. In other words, reference frames that move at constant velocity with respect to each other are also inertial where Newton's second law in the form $\mathbf{F} = m\mathbf{a}$ holds. Thus the *form* of Newton's second law is *invariant* in inertial frames, and therefore, the physics in different inertial frames is indistinguishable (Galilean principle of relativity).

Comparing (4) and (8), it is also important to note that $F_P$ is the total force on the point particle as observed in both the inertial ($\mathcal{I}$) and the arbitrary translating frame ($\mathcal{R}$). There really is no other force on $P$. However, looking at the right-hand side of (8), we see the appearance of the term $m_P{}^{\mathcal{I}}\mathbf{a}_{O'/O}$, which traditionally is called a fictitious force in the literature. As the reader can trace back, the origin of this term is kinematic, and is rooted in the acceleration addition formula (7), at which point no forces, and hence, dynamics, have entered the picture. Therefore, the term $m_P{}^{\mathcal{I}}\mathbf{a}_{O'/O}$, or the so-called fictitious force, is purely a kinematic (relative motion) effect devoid of dynamical origins. The reader may realize the additional advantage of the approach (and the notation adopted), in that one does not have to know in advance what the fictitious forces are in a given situation but let them emerge naturally through the associated kinematics. Generally, in mechanics, identifying the true forces ($F_P$) acting on a particle is difficult enough.[8] Therefore, the analysis of motion is further burdened if one is required to know, a priori, what fictitious forces need to be accounted for (in a dynamical sense) instead of making them emerge as part of kinematics.

We next apply these learnings to the situation where a ball is thrown vertically upward in a rectilinearly moving train. We use this particular scenario to understand, in a concrete way, why the ancients believed the Earth is immovable, and why balls and rocks thrown up do land in our hands.

# 4 A Ball Thrown Vertically Upward from a Moving Train

Our goal here is to utilize the expressions developed so far to analyze the motion of a ball thrown vertically upward from a moving train. We consider the ball to be a point particle and assume that there is no ceiling to the train, and no air resistance or wind to disturb the motion of the ball. We also assume that there is no friction on the train from the rails. Since we expect the ball to reach a height that is negligible compared to the radius of the Earth, we take the gravitational force acting on the ball to be constant; hence, the gravitational acceleration, $g$, is constant.

We consider both the train ($\mathcal{R} \equiv \{O', \mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}\}$) and the station ($\mathcal{I} \equiv \{O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$) to be on the same plane where the former moves rectilinearly relative to the latter in the horizontal $\mathbf{e}_x$ direction (that is, in the positive $x$-direction), where $\mathbf{e}_x = \mathbf{e}_{x'}$. Also, we take $\mathbf{e}_y = \mathbf{e}_{y'}$ direction to be the vertically upward direction (that is, the positive $y$-, or equivalently, the positive $y'$-direction). The station is at rest with respect to the absolute space and, therefore, is inertial. The train is then translating rectilinearly with respect to the inertial station. We assume that time between these two frames is synchronized, and that at time $t = 0$ the origin $O'$ of the passing train coincides with the origin $O$ of the station, at which instant a ball is thrown vertically upwards by an experimenter on the train. As described earlier, once synchronized, all frames of reference read the same time in Newtonian mechanics irrespective of their motion. This is important in our analysis

[8] Historically, this is one of the reasons that led to the development of the Lagrangian, Hamiltonian, and Hertzian formulations of mechanics, where one does not have to explicitly identify the forces acting on particles.

since, once synchronized, the station time is always the same as the train time. This essentially means that the speed of the train is negligible compared to the speed of light, which indeed is the case for earthly trains, including the Earth itself. We also assume the Earth to be an inertial frame, since this is the case for many practical cases as described earlier. We analyze two cases: (a) where the train moves at constant velocity with respect to the station, and (b) where the train moves at constant acceleration with respect to the station.

**(a) Motion of a Ball Thrown Vertically Upward from a Train Moving at Constant Velocity**

Since the train is moving at constant velocity relative to the station, its acceleration with respect to the station vanishes. Therefore, ${}^{\mathcal{I}}\mathbf{a}_{O'/O} = \mathbf{0}$. Thus, (8) reads

$$m_P\,{}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{F}_P. \tag{9}$$

Given the form of this equation, which is Newton's second law, the train moving at constant velocity relative to the stationary station, itself, is an inertial reference frame. For completeness, here, $m_P \neq 0$ is the mass of the ball, $\mathbf{F}_P$ is the total force acting on the ball, and ${}^{\mathcal{R}}\mathbf{a}_{P/O'}$ is the acceleration of the ball as observed on the train.

Let the experimenter on the train throw the ball vertically upward with a speed $v_0$ at time $t = 0$ when the origins of the two frames coincide; therefore, the initial velocity of the ball relative to the train is ${}^{\mathcal{R}}\mathbf{v}_{P/O'}(0) = v_0\mathbf{e}_{y'}$. Then, the position vector of the ball, according to this experimenter, is

$$\mathbf{r}_{P/O'} = x'\mathbf{e}_{x'} + y'\mathbf{e}_{y'}.$$

($z' = 0$ since the motion takes place in the $X' - Y'$ plane of the train, which is the same as the $X - Y$ plane of the station.) Therefore, the ball's velocity and acceleration as observed in the train are given by

$${}^{\mathcal{R}}\mathbf{v}_{P/O'} = \dot{x}'\mathbf{e}_{x'} + \dot{y}'\mathbf{e}_{y'}, \quad {}^{\mathcal{R}}\mathbf{a}_{P/O'} = \ddot{x}'\mathbf{e}_{x'} + \ddot{y}'\mathbf{e}_{y'}.$$

Now, the only force acting on the ball is the gravitational force from the Earth which acts vertically downward with constant magnitude $m_P g$. Therefore, $\mathbf{F}_P = -m_P g\mathbf{e}_{y'}$. Substituting these expressions for force and acceleration in (9) and dividing the resulting equation by $m_P$, we obtain

$$\ddot{x}'\mathbf{e}_{x'} + \ddot{y}'\mathbf{e}_{y'} = -g\mathbf{e}_{y'}.$$

We can now equate the coefficients of the corresponding unit vectors to obtain the scalar equations of motion (EOM). Thus, the equations of motion of the ball as expressed in the train frame, which is inertial, are

$$\boxed{\ddot{x}' = 0, \quad \ddot{y}' = -g. \quad \text{EOM of the ball as observed on the inertial train}} \tag{10}$$

Note that in these equations of motion there is no information regarding the constant speed of the train relative to the station. Therefore, the description of the motion of the ball with respect to the uniformly moving train is independent of how fast the train is moving relative to the station.

Our goal now is to solve these equations of motion. The equation $\ddot{x}' = 0$ implies that the ball does not accelerate in the horizontal $\mathbf{e}_{x'}$ direction. In other words, there is no acceleration of the ball along the length of the train as measured by the observers on the train. This immediately implies that the ball's horizontal speed, $\dot{x}'$, as observed in the train, must remain a constant throughout its motion, including that at time $t = 0$. But, since the ball was thrown vertically upward in the train, there is no horizontal speed given to it. Thus, $\dot{x}'(0) = 0$, and therefore, $\dot{x}'(t) = 0,\ \forall t$. This in turn implies that the coordinate $x'$ of the ball, as observed in the train, must be a constant throughout its motion, including that at time $t = 0$. But, since the ball was initially thrown from the origin $O'$ of the train reference frame where $x' = y' = 0$, this implies that $x'(0) = 0$, and therefore, $x'(t) = 0,\ \forall t$. Hence, there is no lateral movement to the ball when it is thrown vertically upward from a train moving at constant velocity relative to the station; the ball will therefore land at the same spot from which it was launched!

Regarding the above conclusion, the reader may exclaim "of course, we know that!"; yet, this is a profound result. We should now be able to appreciate, in a rigorous way, why the ancients believed that the Earth is immovable, and why it required great strides in our understanding of motion to realize that the Earth can indeed move without leaving the things thrown upward behind. As we have described earlier, the Earth's motion about its axis and around the Sun is such that the acceleration experienced by earthlings due to its motion is negligible. Therefore, the Earth is practically an inertial frame of reference for short-duration motions experienced on a daily basis. As a result, we are able to catch a rock thrown vertically upward without requiring any lateral movement since the rock effectively remains directly above us; its motion effectively only differs in height although the Earth we are standing on is moving at fantastic speeds with respect to absolute space.

Having solved the mystery of the absence of any horizontal movement of the ball, let us consider the remaining equation of motion which describes its vertical movement. Since $\ddot{y}' = -g$, this implies that $\frac{d\dot{y}'}{dt} = -g$. As described above, at $t = 0$, $\dot{y}'(0) = v_0$. Therefore,

$$\int_{v_0}^{\dot{y}'} d\dot{y}' = -g \int_0^t dt \implies \dot{y}' = v_0 - gt.$$

Since $\dot{y}' = \frac{dy'}{dt}$,

$$\int_0^{y'} dy' = \int_0^t (v_0 - gt)\, dt \implies y'(t) = v_0 t - \frac{1}{2} g t^2.$$

Let us summarize these findings:

**(a.1) Motion of the ball according to the train moving with constant velocity**

At $t = 0$ a ball ($P$) is thrown vertically upward from the origin [$O' = (0,0)_{\mathcal{R}}$] in the train ($\mathcal{R}$), which moves with constant velocity relative to the station ($\mathcal{I}$). The ball's initial velocity in the train frame is ${}^{\mathcal{R}}\mathbf{v}_{P/O'}(0) = v_0 \mathbf{e}_{y'}$. The origin [$O = (0,0)_{\mathcal{I}}$] of the station coincides with the origin of the train at $t = 0$. The motion of the ball with respect to the inertial train is then described by:

EOM: $\ddot{x}' = 0$, $\ddot{y}' = -g$, where $x'(0) = 0$ and $\dot{x}'(0) = 0$; $y'(0) = 0$ and $\dot{y}'(0) = v_0$.

Therefore,

$x'(t) = 0 \ \ \forall t, \quad y'(t) = v_0 t - \frac{1}{2} g t^2; \quad v_0 \neq 0.$

Hence, the trajectory of the ball with respect to the train is a vertical straight line. The description of the motion of the ball with respect to the uniformly moving train is independent of how fast the train is moving relative to the station.

When $y'(t) = 0$, $v_0 t - \frac{1}{2} g t^2 = 0$ implies that $t(v_0 - \frac{1}{2} g t) = 0$. Therefore, $t = 0$ and $t = 2v_0/g$ are solutions. The former corresponds to the time of launch of the ball, and the latter corresponds to the time of its arrival back at the launch point ($O'$). Therefore, the ball is in flight for a duration of $T = 2v_0/g$ and it will reach the zenith of its trajectory at $\tau = T/2 = v_0/g$; hence, the maximum height the ball reaches in the train frame is $y'(\tau) = y'_{\max} = v_0 \tau - \frac{1}{2} g \tau^2 = \frac{v_0^2}{2g}$.

We now turn to describing the motion of the ball with respect to the station. In analyzing the motion of the ball in the train we have not cared to specify the details of motion of the train other than to state the fact that it is moving at constant velocity relative to the station. In order to describe the motion of the ball with respect to the station, we need to specify to speed and the direction of the train. Therefore, let us take the train to be moving at constant speed $u$ in the positive $\mathbf{e}_x$ direction. Thus the velocity of the train relative to the station is ${}^{\mathcal{I}}\mathbf{v}_{O'/O} = u\mathbf{e}_x$, which is constant. At any instance, the position of the ball according to the station frame is $(x, y)_{\mathcal{I}}$. Therefore its position vector is

$$\mathbf{r}_{P/O} = x\mathbf{e}_x + y\mathbf{e}_y.$$

($z = 0$ since the motion takes place in the $X - Y$ plane of the station, which is the same as the $X' - Y'$ plane of the train.) Therefore, the ball's velocity and acceleration as observed in the station are given by

$$^{\mathcal{I}}\mathbf{v}_{P/O} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y, \;\; ^{\mathcal{I}}\mathbf{a}_{P/O} = \ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y.$$

Now, from (6), $^{\mathcal{I}}\mathbf{v}_{P/O} = {}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\mathbf{v}_{O'/O}$. Therefore, at the moment the ball is vertically launched,

$$^{\mathcal{I}}\mathbf{v}_{P/O}(0) = {}^{\mathcal{R}}\mathbf{v}_{P/O'}(0) + {}^{\mathcal{I}}\mathbf{v}_{O'/O}(0).$$

But $^{\mathcal{R}}\mathbf{v}_{P/O'}(0) = v_0\mathbf{e}_{y'} = v_0\mathbf{e}_y$ and $^{\mathcal{I}}\mathbf{v}_{O'/O}(0) = u\mathbf{e}_x$, where we have used the fact that $\mathbf{e}_{y'} = \mathbf{e}_y$. Thus,

$$^{\mathcal{I}}\mathbf{v}_{P/O}(0) = v_0\mathbf{e}_y + u\mathbf{e}_x = \dot{x}(0)\mathbf{e}_x + \dot{y}(0)\mathbf{e}_y \implies \dot{x}(0) = u, \;\; \dot{y}(0) = v_0.$$

Since the station frame is inertial, from Newton's second law,

$$m_P\,{}^{\mathcal{I}}\mathbf{a}_{P/O} = \mathbf{F}_P. \tag{11}$$

Now, the only force acting on the ball is the gravitational force from the Earth which acts vertically downward with constant magnitude $m_P g$. Therefore, $\mathbf{F}_P = -m_P g\mathbf{e}_y$. Substituting these expressions for force and acceleration in (11) and dividing the resulting equation by $m_P$, we obtain

$$\ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y = -g\mathbf{e}_y.$$

We can now equate the coefficients of the corresponding unit vectors to obtain the scalar equations of motion (EOM). Thus, the equations of motion of the ball as expressed in the station frame, which is inertial, are

$$\boxed{\ddot{x} = 0, \quad \ddot{y} = -g. \quad \text{EOM of the ball as observed on the (inertial) station}} \tag{12}$$

We therefore note that the equations of motion of the ball as expressed according to the station are identical to the equations of motion expressed according to the train [see (10)]. This must be so since the station is at rest in absolute space and the train is moving at constant velocity relative to the station; hence, they are both inertial frames, and as such, the laws of physics must have the same form in both in describing the same phenomena (Galilean Principle of Relativity). However, this does not mean that the observations made in the two inertial frames are identical since the initial conditions as measured by the respective observers are different: on the train, $x'(0) = 0$, $\dot{x}'(0) = 0$, $y'(0) = 0$, $\dot{y}'(0) = v_0$; but on the station, $x(0) = 0$, $\dot{x}(0) = u$, $y(0) = 0$, $\dot{y}(0) = v_0$.

Now, since $\ddot{x} = 0$, it follows that the horizontal speed of the ball with respect to the station is constant, including at the instance $t = 0$: that is $\dot{x}(t) = \dot{x}(0)$. But, as established above, $\dot{x}(0) = u$. Therefore, $\dot{x} = u$, $\forall t$. Now, since the ball is launched when the origin of the train coincides with the origin of the station, $x = y = 0$ at $t = 0$. Hence,

$$\int_0^x dx = u \int_0^t dt \implies x = ut.$$

The remaining equation of motion describes vertical movement of the ball with respect to the station. Since $\ddot{y} = -g$, this implies that $\frac{d\dot{y}}{dt} = -g$. As shown above, at $t = 0$, $\dot{y}(0) = v_0$. Therefore,

$$\int_{v_0}^{\dot{y}} d\dot{y} = -g \int_0^t dt \implies \dot{y} = v_0 - gt.$$

Since $\dot{y} = \frac{dy}{dt}$,

$$\int_0^y dy = \int_0^t (v_0 - gt)\, dt \implies y(t) = v_0 t - \frac{1}{2} g t^2.$$

Thus, both the station and the train frame describe the vertical movement of the ball identically; that is, $y(t) = y'(t)$. Now, since $x = ut$, $t = x/u$. Substituting this in the last expression for $y$ to eliminate time yields

$$y = \frac{v_0}{u} x - \frac{g}{2u^2} x^2,$$

which is the equation of a parabola. Let us summarize these findings:

**(a.2) Motion of the ball according to the station (train moving with constant velocity)**

At $t = 0$ a ball ($P$) is thrown vertically upward from the origin $[O' = (0,0)_{\mathcal{R}}]$ in the train ($\mathcal{R}$), which moves with constant velocity ${}^{\mathcal{I}}\mathbf{v}_{O'/O} = u\mathbf{e}_x$ relative to the station ($\mathcal{I}$). The ball's initial velocity in the train frame is ${}^{\mathcal{R}}\mathbf{v}_{P/O'}(0) = v_0\mathbf{e}_{y'}$. The origin $[O = (0,0)_{\mathcal{I}}]$ of the station coincides with the origin of the train at $t = 0$. The motion of the ball with respect to the inertial station is then described by:

EOM: $\ddot{x} = 0,\ \ \ddot{y} = -g$, where $x(0) = 0$ and $\dot{x}(0) = u$; $y(0) = 0$ and $\dot{y}(0) = v_0$.

Therefore,

$$x(t) = ut, \quad y(t) = v_0 t - \tfrac{1}{2}gt^2 = y'(t) \quad \Longrightarrow \quad y = \tfrac{v_0}{u}x - \tfrac{g}{2u^2}x^2; \quad u, v_0 \neq 0$$

Hence, the trajectory of the ball with respect to the station is a parabola.

Just as observed on the train, since $y(t) = v_0 t - \frac{1}{2}gt^2$, the ball reaches the launch point $O'$ at time $T = 2v_0/g$, which at that instance is a distance $x = uT = 2uv_0/g$ away from the origin $O$ of the station. As observed in the station, when the ball reaches its highest point at time $\tau = T/2 = v_0/g$, its coordinates read $(x,y)_{\mathcal{I}}|_{t=\tau} = \left(\frac{uv_0}{g}, \frac{v_0^2}{2g}\right)$. In contrast, in the train frame, when the ball reaches the highest point, its coordinates read $(x',y')_{\mathcal{R}}|_{t=\tau} = \left(0, \frac{v_0^2}{2g}\right)$. Figure 3 shows the trajectory of the ball in the respective frames.

If an identical experiment is conducted on the train station – that is, a ball thrown vertically upward at speed $v_0$ – then all the observations made by the observer on the station will agree with those made on the train, including the key observation that the ball lands in the hands of the thrower without needing any lateral movement. Therefore, if the observers on the station and on the train have no means to look outside, they will not be able to distinguish, via any experiment conducted within their enclosures, whether they are at rest or are in uniform motion. Identical experiments conducted in inertial frames yield identical results as Galileo's principle of relativity states. We now turn to the scenario where the train moves at constant acceleration relative to the station.

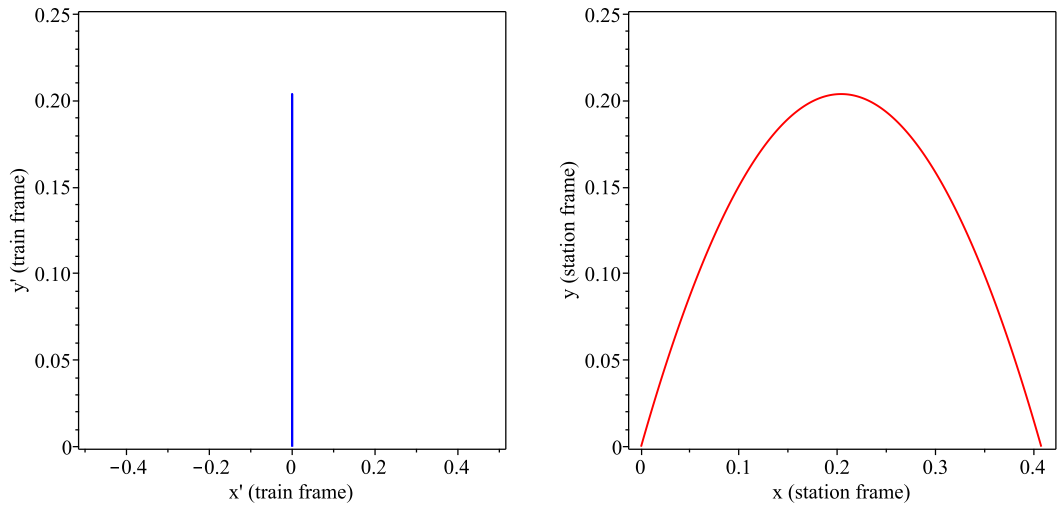


Figure 3: Trajectory of a ball as observed on a uniformly moving train (left, blue) and on the station (right, red) during the following experiment: The ball ($P$) was thrown vertically upward at $t = 0$ from the origin $[O' = (0,0)_{\mathcal{R}}]$ in the train ($\mathcal{R}$), which moves at constant velocity ${}^{\mathcal{I}}\mathbf{v}_{O'/O} = u\mathbf{e}_x$ relative to the station ($\mathcal{I}$), where $u = 1$ m/s. The ball's initial velocity in the train frame is ${}^{\mathcal{R}}\mathbf{v}_{P/O'}(0) = v_0\mathbf{e}_{y'}$, where $v_0 = 2$ m/s. The origin $[O = (0,0)_{\mathcal{I}}]$ of the station coincides with the origin $O'$ of the train at $t = 0$; $g = 9.81$ m/s$^2$. All distances shown in the charts are in meters.

**(b) Motion of a Ball Thrown Vertically Upward from a Train Moving with Constant Acceleration**

Let the constant acceleration of the train relative to the station be ${}^{\mathcal{I}}\mathbf{a}_{O'/O} = a\mathbf{e}_x$, where $a \neq 0$ is a constant. Since $\mathbf{e}_x = \mathbf{e}_{x'}$, it is also true that ${}^{\mathcal{I}}\mathbf{a}_{O'/O} = a\mathbf{e}_{x'}$. Since $a \neq 0$, the train is no longer an inertial frame. Therefore, if applied in the non-inertial train to describe the motion of the ball, Newton's second law reads

$$m_P\,{}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{F}_P - m_P\,{}^{\mathcal{I}}\mathbf{a}_{O'/O}. \tag{13}$$

Using the corresponding expressions stated under the scenario (a) above, it then follows according to (13), that

$$m_P\,{}^{\mathcal{R}}\mathbf{a}_{P/O'} = m_P\left(\ddot{x}'\mathbf{e}_{x'} + \ddot{y}'\mathbf{e}_{y'}\right) = -m_P g\mathbf{e}_{y'} - m_P a\mathbf{e}_{x'}. \tag{14}$$

Hence,

$$\boxed{\ddot{x}' = -a, \quad \ddot{y}' = -g. \quad \text{EOM of the ball as observed on the non-inertial train}} \tag{15}$$

As before, the solutions of these differential equations are straightforward. They yield: $x'(t) = \dot{x}'(0)t - \frac{1}{2}at^2$ and $y'(t) = \dot{y}'(0)t - \frac{1}{2}gt^2$. Since the ball was thrown vertically upward with a velocity ${}^{\mathcal{R}}\mathbf{v}_{P/O'}(0) = v_0\mathbf{e}_{y'}$ at $t = 0$, $\dot{x}'(0) = 0$ and $\dot{y}'(0) = v_0$. Therefore, $x'(t) = -\frac{1}{2}at^2$ and $y'(t) = v_0t - \frac{1}{2}gt^2$. Let us rearrange this last expression to read $v_0t = y' + \frac{1}{2}gt^2$. If we now square this expression we will obtain a bi-quadratic equation in $t^2$, which is: $v_0^2t^2 = y'^2 + gy't^2 + \frac{1}{4}g^2t^4$. Since $x' = -\frac{1}{2}at^2$, it follows that $t^2 = -2x'/a$. Substituting this in the bi-quadratic and multiplying the resultant by $a^2$ followed by rearrangement, we obtain: $g^2x'^2 + a^2y'^2 - 2agx'y' + 2av_0^2x' = 0$, which is a more complex-looking conic section. We summarize these results below:

**(b.1) Motion of the ball according to the train moving with constant acceleration**

At $t = 0$ a ball ($P$) is thrown vertically upward from the origin [$O' = (0,0)_{\mathcal{R}}$] in the train ($\mathcal{R}$), which moves with constant acceleration ${}^{\mathcal{I}}\mathbf{a}_{O'/O} = a\mathbf{e}_x$ relative to the station ($\mathcal{I}$). The ball's initial velocity in the train frame is ${}^{\mathcal{R}}\mathbf{v}_{P/O'}(0) = v_0\mathbf{e}_{y'}$. The origin [$O = (0,0)_{\mathcal{I}}$] of the station coincides with the origin of the train at $t = 0$. The motion of the ball with respect to the non-inertial train is then described by:

EOM: $\ddot{x}' = -a$, $\ddot{y}' = -g$, where $x'(0) = 0$ and $\dot{x}'(0) = 0$; $y'(0) = 0$ and $\dot{y}'(0) = v_0$.

Therefore,

$$x'(t) = -\tfrac{1}{2}at^2, \quad y'(t) = v_0 t - \tfrac{1}{2}gt^2 \quad \Longrightarrow \quad g^2x'^2 + a^2y'^2 - 2agx'y' + 2av_0^2x' = 0; \quad a, v_0 \neq 0.$$

Hence, the trajectory of the ball with respect to the train is a more general conic section.

Note that since we consider the train to be accelerating in the positive $x$-direction, $a > 0$. Therefore, according to $x'(t) = -\frac{1}{2}at^2$, negative $x'$ increases quadratically with time. This means the ball is (increasingly) left behind according to the observers on the train; in other words, as observed in the train, the ball moves increasingly in the negative $x$-direction as the train accelerates in the positive $x$-direction. This effect can be seen on the left chart in Figure 4. From $y'(t) = v_0 t - \frac{1}{2}gt^2$ it is again clear that the ball is in flight for a duration $T = 2v_0/g$. However, when it reaches $y' = 0$ again, it will not land at the origin $O' = (0,0)$; instead, as observed in the train, it will land at $x' = -2av_0^2/g^2$. As observed in the train, when the ball reaches its highest point at time $\tau = T/2 = v_0/g$, its coordinates read $(x', y')_{\mathcal{R}}|_{t=\tau} = \left(-\frac{1}{2}\frac{av_0^2}{g^2}, \frac{v_0^2}{2g}\right)$.

If we revisit (14), since ${}^{\mathcal{R}}\mathbf{a}_{P/O'} = -a\mathbf{e}_{x'} - g\mathbf{e}_{y'}$, we see that the acceleration of the ball as observed in the train is a combination of the (constant) acceleration of the train relative to the station and the (constant) gravitational acceleration of the ball. The former acts in the $-\mathbf{e}_{x'}$ direction (say, west) while the latter acts in the $-\mathbf{e}_{y'}$ direction (say, south). From vector addition, this implies that the resultant acceleration (${}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{g}'$) of the ball with respect to the train has constant magnitude $g' = \sqrt{a^2 + g^2}$, which acts at a constant angle $\theta$ in the south-west direction such that $\tan\theta = a/g$. Therefore, when the ball is thrown vertically upward in the accelerating train, we can consider it to be moving under an effective constant gravitational field $\mathbf{g}'$ that acts at an angle $\theta$ to the downward vertical in the direction opposite to the moving train. The more general conic-section trajectory results from this effective gravitational field.

To complete the story, let us look at the motion of the ball with respect to the station when it is thrown vertically upward from the accelerating train. Since the station remains inertial, the motion of the ball is described according to the Newton's second law: $m_P {}^{\mathcal{I}}\mathbf{a}_{P/O} = \mathbf{F}_P$. Therefore, the resulting conclusions are the same as before when the ball was thrown vertically upward from the train moving at constant velocity relative to the station. In the present case, let us assume that when the origins of the frames coincide at $t = 0$, that is, at the instance that the ball is launched, the velocity of the train relative to the station is ${}^{\mathcal{I}}\mathbf{v}_{O'/O}(0) = u_0\mathbf{e}_x$. We can then state the following summary:

**(b.2) Motion of the ball according to the station (train moving with constant acceleration**

At $t = 0$ a ball ($P$) is thrown vertically upward from the origin $[O' = (0,0)_{\mathcal{R}}]$ in the train ($\mathcal{R}$), which moves with constant acceleration ${}^{\mathcal{I}}\mathbf{a}_{O'/O} = a\mathbf{e}_x$ relative to the station ($\mathcal{I}$). The ball's initial velocity in the train frame is ${}^{\mathcal{R}}\mathbf{v}_{P/O'}(0) = v_0\mathbf{e}_{y'}$. The origin $[O = (0,0)_{\mathcal{I}}]$ of the station coincides with the origin of the train at $t = 0$ at which instance the velocity of the train relative to the station is ${}^{\mathcal{I}}\mathbf{v}_{O'/O}(0) = u_0\mathbf{e}_x$. The motion of the ball with respect to the inertial station is then described by:

EOM: $\ddot{x} = 0, \quad \ddot{y} = -g$, where $x(0) = 0$ and $\dot{x}(0) = u_0$; $y(0) = 0$ and $\dot{y}(0) = v_0$.

Therefore,

$$x(t) = u_0 t, \quad y(t) = v_0 t - \tfrac{1}{2}gt^2 = y'(t) \quad \Longrightarrow \quad y = \tfrac{v_0}{u_0}x - \tfrac{g}{2u_0^2}x^2; \quad u_0, v_0 \neq 0$$

Hence, the trajectory of the ball with respect to the station is a parabola.

As can be seen, since $y(t) = y'(t)$, the vertical description of the motion between the two frames is the same. However, in contrast to the uniformly moving train where the equations of motion were identical in the two inertial frames, the equations of motion differ between the two frames when the train is accelerating. In this case, the accelerating train is a non-inertial frame while the station remains an inertial frame. The ball will be in flight for a duration of $T = 2v_0/g$ and will land at $x = u_0 T = 2u_0 v_0/g$ as observed in the station. Figure 4 shows the trajectory of the ball in the respective frames.

If the ball is hanging via a taut string attached to the ceiling of the accelerating train, then due to the effective gravitational field on the ball, the string will make an angle $\theta$ to the downward vertical in the direction opposite to the moving train such that $\tan\theta = a/g$. This provides our observer on the train another means of determining the non-inertial status of the accelerating train. In contrast, if the train is moving uniformly relative to the station, then $a = 0$, and hence, $\theta = 0$. Therefore, the string will hang vertically down from the ceiling of a uniformly moving train; the same observation can be made by the observers on the station via an identical experiment conducted in their frame. Thus, inertial and non-inertial frames can be distinguished via identical mechanical experiments.

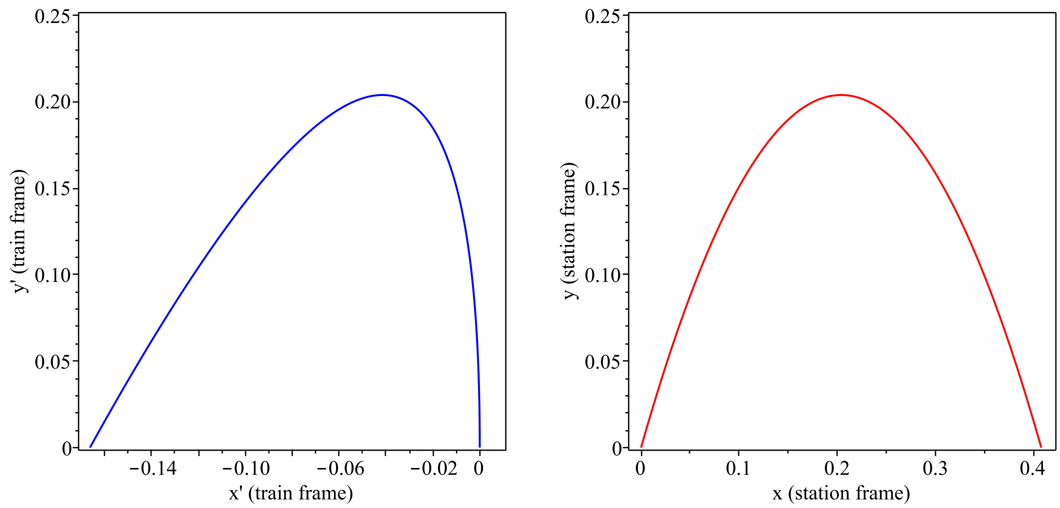


Figure 4: Trajectory of a ball as observed on an accelerating train (left, blue) and on the station (right, red) during the following experiment: The ball ($P$) was thrown vertically upward at $t = 0$ from the origin [$O' = (0, 0)_{\mathcal{R}}$] in the train ($\mathcal{R}$), which moves with constant acceleration ${}^{\mathcal{I}}\mathbf{a}_{O'/O} = a\mathbf{e}_x$ relative to the station ($\mathcal{I}$), where $a = 2$ m/s$^2$. The ball's initial velocity in the train frame is ${}^{\mathcal{R}}\mathbf{v}_{P/O'}(0) = v_0\mathbf{e}_{y'}$, where $v_0 = 2$ m/s. The origin [$O = (0, 0)_{\mathcal{I}}$] of the station coincides with the origin $O'$ of the train at $t = 0$ at which instance ${}^{\mathcal{I}}\mathbf{v}_{O'/O} = u_0\mathbf{e}_x$, where $u_0 = 1$ m/s; $g = 9.81$ m/s$^2$. All distances shown in the charts are in meters.

# 5 Newton's Second Law: Translations & Rotations

We now consider a reference frame $\mathcal{R}$ that is both translating and rotating relative to an inertial frame $\mathcal{I}$. We therefore wish to describe the motion of a point particle $P$ with respect this translating and rotating frame. Since this scenario is more general, the equations (6) and (8) that apply only for translations must emerge as special cases of equations that apply under both translation and rotation. Therefore, when frame $\mathcal{R}$ is rotating in addition to translation, we expect the formulae to read:

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{v}_{P/O} &= {}^{\mathcal{I}}\mathbf{v}_{O'/O} + {}^{\mathcal{R}}\mathbf{v}_{P/O'} + [\text{rotation-dependent terms}], \\ m_P\,{}^{\mathcal{R}}\mathbf{a}_{P/O'} &= F_P - m_P\,{}^{\mathcal{I}}\mathbf{a}_{O'/O} + [\text{rotation-dependent terms}]. \end{aligned}$$

Our goal, therefore, is to identify the structure of these extra rotation-dependent terms. Toward this end, we again confine ourselves to the planar case. First let us assume that frame $\mathcal{R}$ is rotating relative to the inertial frame $\mathcal{I}$ about an axis perpendicular to the plane that goes through its origin $O'$. Let us denote the angular velocity of frame $\mathcal{R}$ relative to the inertial frame $\mathcal{I}$ as ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$. Since the axis about which $\mathcal{R}$ rotates is parallel to $\mathbf{e}_{z'}$ $(= \mathbf{e}_z)$, the angular velocity is then

$$ {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\mathbf{e}_{z'} = \omega\mathbf{e}_z\ , \tag{16}$$

where $\omega$ is the magnitude of ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$. As a second step toward determining the structure of the rotation-dependent terms, we define a relation which is fundamental in analyzing motion. In order to motivate this definition, note from (5) that, under pure translation,

$$ \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O'} = \frac{{}^{\mathcal{R}}d}{dt}\mathbf{r}_{P/O'}. \tag{17}$$

When a rotation is added to frame $\mathcal{R}$ relative to $\mathcal{I}$, we expect a term that involves the angular velocity ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$ to appear on the right-hand side of the above expression. Since both sides of (17) are vectors (measured in length per unit time), the only way we can add ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$ (which is measured in per unit time) to the right of (17) is to form a vector cross product of ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$ with a vector having the units of length. (Note that a vector dot product with ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$ would not do since this would result in a scalar while the other terms are vectors; a scalar cannot be added to a vector.) Therefore, it seems reasonable that we form the vector cross product ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}$ to be added to (17). We therefore have,

$$ \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O'} = \frac{{}^{\mathcal{R}}d}{dt}\mathbf{r}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\ . \tag{18}$$

When $\mathcal{R}$ is only translating, ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \mathbf{0}$, and we recover (17) back again. If we extract only the operator part from (18), we then have,

$$\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O'} = \left(\frac{{}^{\mathcal{R}}d}{dt} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}\times\right)\mathbf{r}_{P/O'} \implies \frac{{}^{\mathcal{I}}d}{dt} \equiv \left(\frac{{}^{\mathcal{R}}d}{dt} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}\times\right).$$

The operator form indicates that there is nothing special about $\mathbf{r}_{P/O'}$ and that we can apply it to any vector. Moreover, $\mathcal{I}$ and $\mathcal{R}$ can also be considered as arbitrary frames. This, then, results in a very general and fundamental expression known as the Transport Equation, which we state below.

**Transport Equation: definition**

If $\mathcal{A}$ and $\mathcal{B}$ are reference frames, and frame $\mathcal{B}$ rotates relative to frame $\mathcal{A}$ with angular velocity ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}$, then the following is satisfied for any vector $\mathbf{c}$ observed in the respective frames:

$$\frac{{}^{\mathcal{A}}d}{dt}\mathbf{c} = \frac{{}^{\mathcal{B}}d}{dt}\mathbf{c} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{c}. \tag{19}$$

We are now in a position to understand the structure of the rotation-dependent terms. Consider Figure 5.

Starting with the position vector triangle $OPO'$, then,

$$\begin{aligned}
\mathbf{r}_{P/O} &= \mathbf{r}_{O'/O} + \mathbf{r}_{P/O'}\,, \\
\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} &= \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{O'/O} + \left[\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O'}\right]\,, \\
\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} &= \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{O'/O} + \left[\frac{{}^{\mathcal{R}}d}{dt}\mathbf{r}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right]\,, \text{ by applying the transport eq. (19)}
\end{aligned}$$

Now, $\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{P/O}$, $\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{O'/O} = {}^{\mathcal{I}}\mathbf{v}_{O'/O}$, and $\frac{{}^{\mathcal{R}}d}{dt}\mathbf{r}_{P/O'} = {}^{\mathcal{R}}\mathbf{v}_{P/O'}$. Therefore,

$${}^{\mathcal{I}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{O'/O} + \left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right]\,.$$

We now start by taking the time derivative of the last expression with respect to the inertial frame $\mathcal{I}$.

$$\frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\mathbf{v}_{P/O} = \frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\mathbf{v}_{O'/O} + \frac{{}^{\mathcal{I}}d}{dt}\left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right].$$

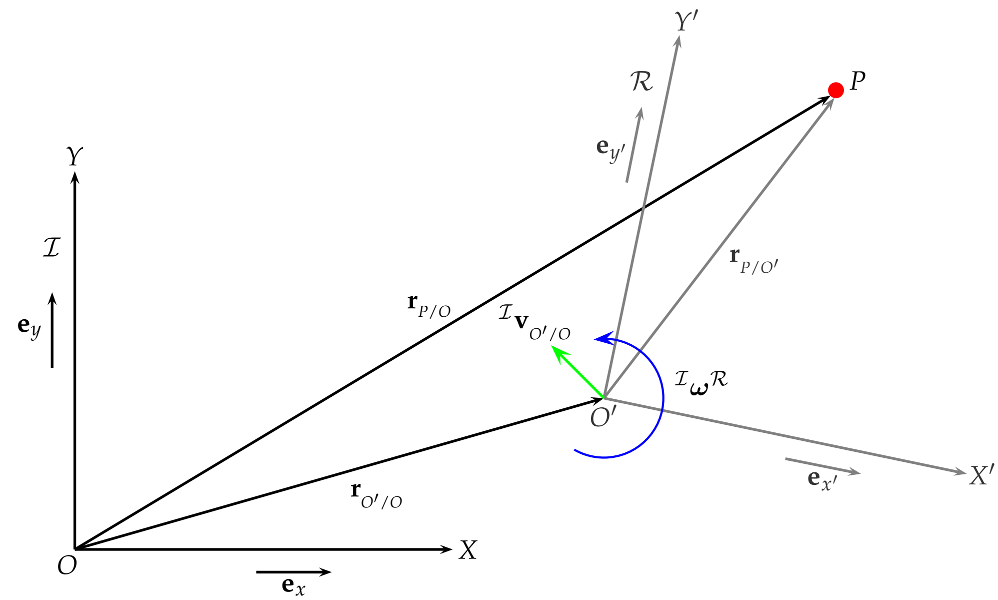


Figure 5: The reference frame $\mathcal{R} \equiv \{O', \mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}\}$ is both translating and rotating relative to the inertial reference frame $\mathcal{I} \equiv \{O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. The green arrow shows the instantaneous translational velocity ($^{\mathcal{I}}\mathbf{v}_{O'/O}$) of $\mathcal{R}$ relative to $\mathcal{I}$. The blue arrow depicts the instantaneous rotation of $\mathcal{R}$ about an axis that passes through its origin perpendicular to the plane; the instantaneous angular velocity of $\mathcal{R}$ relative to $\mathcal{I}$ is $^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$. | Drawing by AD.

Now, $\frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{a}_{P/O}$ and $\frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\mathbf{v}_{O'/O} = {}^{\mathcal{I}}\mathbf{a}_{O'/O}$. Therefore,

$${}^{\mathcal{I}}\mathbf{a}_{P/O} = {}^{\mathcal{I}}\mathbf{a}_{O'/O} + \frac{{}^{\mathcal{I}}d}{dt}\left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right].$$

From the transport equation,

$$\frac{{}^{\mathcal{I}}d}{dt}\left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right] = \frac{{}^{\mathcal{R}}d}{dt}\left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right] + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right].$$

The second term on the right can be easily expanded to read:

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right] = {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right).$$

The first term on the right reads:

$$\frac{{}^{\mathcal{R}}d}{dt}\left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right] = \frac{{}^{\mathcal{R}}d}{dt}{}^{\mathcal{R}}\mathbf{v}_{P/O'} + \left(\frac{{}^{\mathcal{R}}d}{dt}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}\right) \times \mathbf{r}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left(\frac{{}^{\mathcal{R}}d}{dt}\mathbf{r}_{P/O'}\right).$$

Since $\frac{{}^{\mathcal{R}}d}{dt}{}^{\mathcal{R}}\mathbf{v}_{P/O'} = {}^{\mathcal{R}}\mathbf{a}_{P/O'}$ and $\frac{{}^{\mathcal{R}}d}{dt}\mathbf{r}_{P/O'} = {}^{\mathcal{R}}\mathbf{v}_{P/O'}$,

$$\frac{{}^{\mathcal{R}}d}{dt}\left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right] = {}^{\mathcal{R}}\mathbf{a}_{P/O'} + \left(\frac{{}^{\mathcal{R}}d}{dt}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}\right) \times \mathbf{r}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'}.$$

We are now left to find out what the middle term on the right is. It is the time derivative of the angular velocity taken in the rotating frame itself. Considering (16), which is ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\mathbf{e}_{z'} = \omega\mathbf{e}_z$, it follows that

$$
\begin{aligned}
\frac{{}^{\mathcal{R}}d}{dt}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} &= \frac{{}^{\mathcal{R}}d}{dt}\left(\omega \mathbf{e}_{z'}\right), \\
&= \left(\frac{{}^{\mathcal{R}}d}{dt}\omega\right)\mathbf{e}_{z'}, \quad \because\ \mathbf{e}_{z'} \text{ is fixed in } \mathcal{R} \\
&= \left(\frac{{}^{\mathcal{I}}d}{dt}\omega\right)\mathbf{e}_{z'}, \quad \because\ \omega \text{ is a scalar} \\
&= \left(\frac{{}^{\mathcal{I}}d}{dt}\omega\right)\mathbf{e}_{z}, \quad \because\ \mathbf{e}_{z'} = \mathbf{e}_z \\
&= \frac{{}^{\mathcal{I}}d}{dt}\left(\omega \mathbf{e}_{z}\right), \quad \because\ \mathbf{e}_{z} \text{ is fixed in } \mathcal{I} \\
\therefore\ \frac{{}^{\mathcal{R}}d}{dt}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} &= \frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}.
\end{aligned}
$$

Thus the rate of change of angular velocity with respect to the two frames are the same. Specifically, the rate of change of angular velocity with respect to the inertial frame, that is, $\frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$, is the angular acceleration ($\boldsymbol{\alpha}$) of $\mathcal{R}$ relative to the inertial frame $\mathcal{I}$. Hence,

$$
\frac{{}^{\mathcal{R}}d}{dt}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \frac{{}^{\mathcal{I}}d}{dt}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}}. \tag{20}
$$

Therefore,

$$
\frac{{}^{\mathcal{R}}d}{dt}\left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right] = {}^{\mathcal{R}}\mathbf{a}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}} \times \mathbf{r}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'}\,.
$$

Collecting the above results, we then have,

$$
\begin{aligned}
\frac{{}^{\mathcal{I}}d}{dt}\left[{}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right] &= {}^{\mathcal{R}}\mathbf{a}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}} \times \mathbf{r}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right)\,, \\
&= {}^{\mathcal{R}}\mathbf{a}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}} \times \mathbf{r}_{P/O'} + 2{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right).
\end{aligned}
$$

Therefore,

$$
{}^{\mathcal{I}}\mathbf{a}_{P/O} = {}^{\mathcal{I}}\mathbf{a}_{O'/O} + {}^{\mathcal{R}}\mathbf{a}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}} \times \mathbf{r}_{P/O'} + 2{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right).
$$

As before, we note that these are vector equations that are then valid beyond the planar case. We summarize these general velocity and acceleration addition formulae below.

**Newtonian Velocity & Acceleration Addition for Translations and Rotations**

If $\mathcal{I}$ is an inertial reference frame with origin $O$ and $\mathcal{R}$ is a reference frame with origin $O'$ that is translating and rotating relative to $\mathcal{I}$, and $P$ is a point particle, then

$$^{\mathcal{I}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{O'/O} + {}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\,, \tag{21}$$

$$^{\mathcal{I}}\mathbf{a}_{P/O} = {}^{\mathcal{I}}\mathbf{a}_{O'/O} + {}^{\mathcal{R}}\mathbf{a}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}} \times \mathbf{r}_{P/O'} + 2\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right), \tag{22}$$

where

${}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}} \times \mathbf{r}_{P/O'}$ : Euler acceleration
$2\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'}$ : Coriolis acceleration
${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right)$ : centripetal acceleration

The structure of the rotation-dependent terms that we set out to find is now clear. When rotations are present, the velocity addition formula, (6), is augmented by a single term ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}$. In contrast, when rotations are present, the acceleration addition formula, (7), is augmented by three terms: the Euler, Coriolis, and centripetal accelerations. Multiplying (22) by the mass of the particle ($m_P$) followed by rearrangement (along with the fact that $F_P = m_P\,{}^{\mathcal{I}}\mathbf{a}_{P/O}$), we arrive at the form of Newton's second law according to frame $\mathcal{R}$:

**Newton's Second Law in a Translating and Rotating Reference Frame**

If $\mathcal{I}$ is an inertial reference frame with origin $O$ and $\mathcal{R}$ is a reference frame with origin $O'$ that is translating and rotating relative to $\mathcal{I}$, and $P$ is a point particle, then Newton's second law in the translating and rotating frame $\mathcal{R}$ reads

$$m_P\,{}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{F}_P - m_P\,{}^{\mathcal{I}}\mathbf{a}_{O'/O} - m_P\,{}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}} \times \mathbf{r}_{P/O'} - 2m_P\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} - m_P\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right). \tag{23}$$

In addition to the term ${}^{\mathcal{I}}\mathbf{a}_{O'/O}$, we can trace back the emergence of the Euler $\left({}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right)$, Coriolis $\left(2\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'}\right)$, and centripetal $\left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right)\right]$ acceleration terms in our derivation to their kinematic origins. In other words, they arise in our attempt to describe motion with respect to a frame of reference that is non-inertial; the term $\mathbf{F}_P$ alone contains the information about the true forces acting on the particle. In literature, therefore, these acceleration terms (when multiplied by the mass of the particle) are called fictitious forces. The notation we have adopted should help the reader to keep track of the details of these so-called fictitious forces at all times freeing their working memory for other important tasks in analyzing problems of motion.

# 6 Motion of a Ball as Observed on Earth

In our analysis of a ball thrown from a rectilinearly moving train, we assumed the Earth to be an inertial frame. We relax this assumption in the present section, our goal being to understand what effect the Earth's rotation about its axis has on the motion of a ball (which we take to be a particle, as before) as observed by someone standing on Earth's surface. We still disregard Earth's motion around the Sun since its orbital acceleration is five times smaller than its rotational acceleration about its axis. Our interest is in finding where a ball would land under two scenarios: (c) when a ball is dropped from a tower, and (d) when a ball is thrown vertically upward.

**(c) Motion of a Ball Dropped from a Tower**

Let us call the reference frame $\mathcal{R}$ of our earthling who carries out the experiments, the lab frame, whose origin ($O'$) we take to be located at latitude $\lambda$. It is convenient to consider the inertial frame, $\mathcal{I}$, relative to which the lab frame rotates, as located at the center of the Earth. Therefore, the origin ($O$) of $\mathcal{I}$ coincides with the center of the Earth and its associated orthogonal unit vector triad ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) is fixed in absolute space such that $\mathbf{e}_z$ points along the Earth's axis of rotation. Thus, the constant angular velocity of the Earth is ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \omega \mathbf{e}_z$. We can situate the orthogonal unit vector triad of the lab frame such that $\mathbf{e}_{x'}$ points east, $\mathbf{e}_{y'}$ points north, and $\mathbf{e}_{z'}$ points upward along the radial joining $OO'$. All unit vector triads form right-handed coordinate systems. These are depicted in Figure 6.

Now, the motion of a particle with respect to the lab frame is described according to (23). Since the Earth's angular velocity ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}}$ is constant, it follows that ${}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{R}} = \mathbf{0}$. Moreover, since $\omega \approx 7.3 \times 10^{-5}$ $\text{s}^{-1}$, the term ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{P/O'}\right)$ is quadratic in angular speed, which is $\sim 10^{-9}$ $\text{s}^{-2}$. If we couple this with relatively short heights ($\mathbf{r}_{P/O'}$) observed above the Earth, then this whole term can be ignored in the present analysis. Hence, for particles observed in the lab frame on Earth, the Euler and the centripetal accelerations do not matter. The governing equation of motion for particles observed in the lab frame then reduces to

$$m_P \, {}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{F}_P - m_P \, {}^{\mathcal{I}}\mathbf{a}_{O'/O} - 2m_P \, {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} \,. \tag{24}$$

We now embark to write the remaining accelerations in terms of the coordinates employed in the lab frame (since these are the coordinates used by our experimenter). Toward this end we must find how the unit vector triads in the respective frames are related to each other, which is key to analyzing relative motion. (In the case where the ball was thrown from the rectilinearly moving train, we did this analysis without much difficulty since the corresponding unit vectors in both the station and the train frames were parallel to each other.)

To find how the unit vector triads are related it is best to have them in one location. Therefore, without disturbing the orientation, let us move (in absolute space) the lab frame triad to the cen-

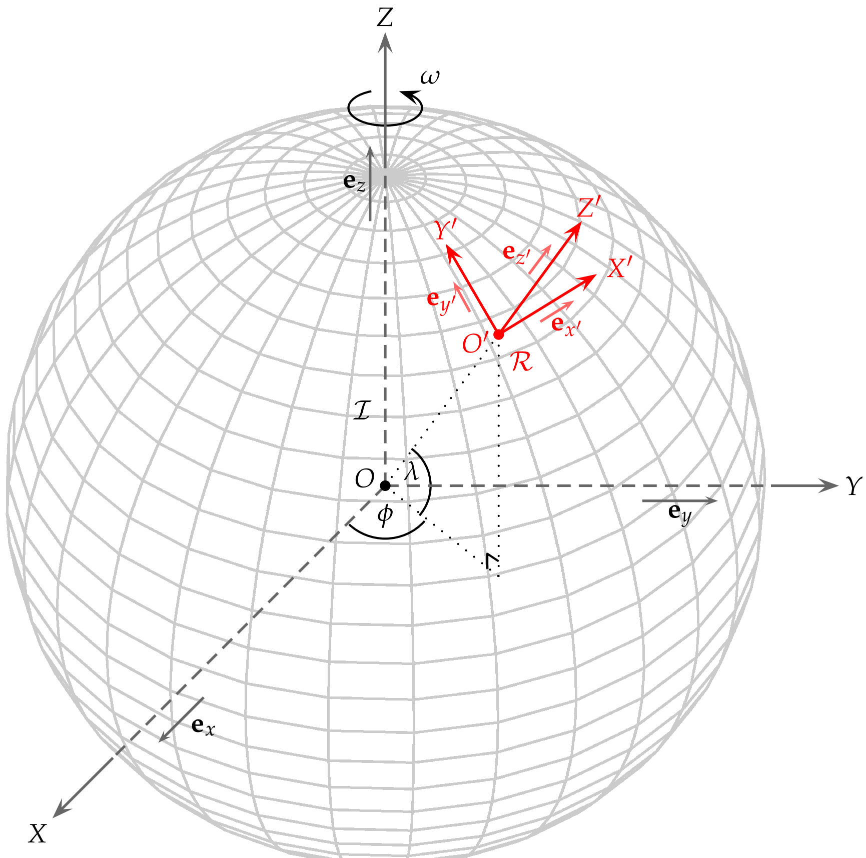


Figure 6: The inertial frame $\mathcal{I} \equiv \{O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is located at the center of the Earth and is fixed in absolute space. The lab frame $\mathcal{R} \equiv \{O', \mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}\}$ on the surface is located at latitude $\lambda$, and at the instant shown, at azimuth $\phi$ (which lies on the $X - Y$ or the equatorial plane). The orthogonal unit vector triad of the lab frame is oriented such that $\mathbf{e}_{x'}$ points east, $\mathbf{e}_{y'}$ points north, and $\mathbf{e}_{z'}$ points directly upward along the radial joining $OO'$. Due to Earth's rotation about its axis, the lab frame rotates at constant angular velocity ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\mathbf{e}_z = \dot{\phi}\mathbf{e}_z$ relative to the inertial frame. | Drawing by AD.

ter of the Earth where the inertial frame triad is located so that the two origins $O'$ and $O$ coincide (see Figure 7).

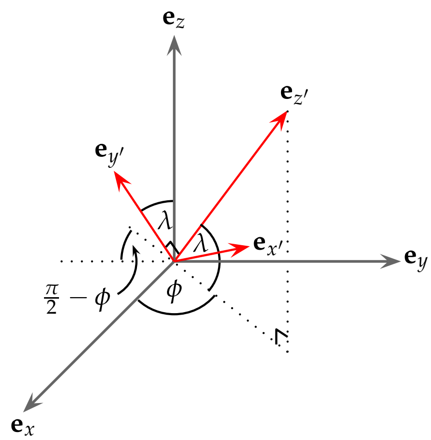


Figure 7: The orthogonal unit vector triad of the lab frame (red) situated at the origin of the inertial frame. The same orientation as in Figure 6 is preserved, which allows us to find the transition table between the two triads. The latitude is $\lambda$, and on the $X-Y$ ($\mathbf{e}_x$–$\mathbf{e}_y$) plane lies both the azimuthal angle $\phi$ and $\mathbf{e}_{x'}$. Note that the unit vectors are not drawn to scale in order to better highlight the orientation and associated angles. | Drawing by AD.

Referring to Figure 7 it can be inferred that $\mathbf{e}_{y'} = -\sin\lambda\cos\phi\mathbf{e}_x - \sin\lambda\sin\phi\mathbf{e}_y + \cos\lambda\mathbf{e}_z$ and $\mathbf{e}_{z'} = \cos\lambda\cos\phi\mathbf{e}_x + \cos\lambda\sin\phi\mathbf{e}_y + \sin\lambda\mathbf{e}_z$. $\mathbf{e}_{x'}$ can then be found by using the fact that $\mathbf{e}_{x'} = \mathbf{e}_{y'} \times \mathbf{e}_{z'}$, which gives $\mathbf{e}_{x'} = -\sin\phi\mathbf{e}_x + \cos\phi\mathbf{e}_y$ (hence, $\mathbf{e}_{x'}$ is on the $X-Y$ plane). We summarize these relations in what is known as a transition table which allows us to write any unit vector in one frame as a combination of unit vectors in another frame.

| | $\mathbf{e}_x$ | $\mathbf{e}_y$ | $\mathbf{e}_z$ |
|---|---|---|---|
| $\mathbf{e}_{x'}$ | $-\sin\phi$ | $\cos\phi$ | 0 |
| $\mathbf{e}_{y'}$ | $-\sin\lambda\cos\phi$ | $-\sin\lambda\sin\phi$ | $\cos\lambda$ |
| $\mathbf{e}_{z'}$ | $\cos\lambda\cos\phi$ | $\cos\lambda\sin\phi$ | $\sin\lambda$ |

Table 1: Transition table for the unit vector triads in the inertial $[\mathcal{I} \equiv \{O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}]$ and the lab $[\mathcal{R} \equiv \{O', \mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}\}]$ frames. A unit vector in the lab frame can be written by adding the inertial frame components across the columns in the corresponding row; a unit vector in the inertial frame can be written by adding the lab frame components across the rows in the corresponding column.

We now have all the ingredients necessary to evaluate the terms in (24). Since $P = (x', y', z')_{\mathcal{R}}$,

$\mathbf{r}_{P/O'} = x'\mathbf{e}_{x'} + y'\mathbf{e}_{y'} + z'\mathbf{e}_{z'}$. Therefore, ${}^{\mathcal{R}}\mathbf{v}_{P/O'} = \dot{x}'\mathbf{e}_{x'} + \dot{y}'\mathbf{e}_{y'} + \dot{z}'\mathbf{e}_{z'}$ and ${}^{\mathcal{R}}\mathbf{a}_{P/O'} = \ddot{x}'\mathbf{e}_{x'} + \ddot{y}'\mathbf{e}_{y'} + \ddot{z}'\mathbf{e}_{z'}$. So the velocity and the acceleration vectors of the point particle $P$ with respect to the lab frame are established.

Next, let us consider the total force $F_P$ on the point particle with mass $m_P$. We assume that only the gravitational force acts on the particle, and that the motion takes place near the Earth's surface so that the gravitational acceleration can be considered constant. Moreover, we assume the particle to be released from rest at a point that lies on the radial $OO'$; that is, from a point directly overhead $O'$ in which direction points the unit vector $\mathbf{e}_{z'}$. Therefore, $F_P = -m_P g\mathbf{e}_{z'}$.

Now, ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\mathbf{e}_z$. Since our interest is in the motion of $P$ as observed in the lab frame, we must express $\mathbf{e}_z$ in terms of the lab frame unit vectors. For this, we look up the transition table, from which we read $\mathbf{e}_z = 0\mathbf{e}_{x'} + \cos\lambda\mathbf{e}_{y'} + \sin\lambda\mathbf{e}_{z'} = \cos\lambda\mathbf{e}_{y'} + \sin\lambda\mathbf{e}_{z'}$. Therefore, ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\cos\lambda\mathbf{e}_{y'} + \omega\sin\lambda\mathbf{e}_{z'}$.

Since ${}^{\mathcal{R}}\mathbf{v}_{P/O'} = \dot{x}'\mathbf{e}_{x'} + \dot{y}'\mathbf{e}_{y'} + \dot{z}'\mathbf{e}_{z'}$ and ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\cos\lambda\mathbf{e}_{y'} + \omega\sin\lambda\mathbf{e}_{z'}$, it follows that,

$$ {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} = \left(\dot{z}'\omega\cos\lambda - \dot{y}'\omega\sin\lambda\right)\mathbf{e}_{x'} + \dot{x}'\omega\sin\lambda\mathbf{e}_{y'} - \dot{x}'\omega\cos\lambda\mathbf{e}_{z'}. $$

We now have only ${}^{\mathcal{I}}\mathbf{a}_{O'/O}$ remaining to be established in terms of the lab frame unit vectors. Since the lab is situated on Earth's surface, and $\mathbf{e}_{z'}$ is along the radial $OO'$, $\mathbf{r}_{O'/O} = R\mathbf{e}_{z'}$, where $R$ is the radius of the Earth.[9] Taking the time derivative of $\mathbf{r}_{O'/O}$ twice in the inertial frame must yield ${}^{\mathcal{I}}\mathbf{a}_{O'/O}$ as follows:

[9] We assume the Earth to be spherical, although, in reality, it is a bit flattened at the poles. Newton's prediction that this would be the case and its eventual verification showed the power and accuracy of his mechanics. At the time, there were competing theories that predicted the opposite effect.

$$
\begin{aligned}
\mathbf{r}_{O'/O} &= R\mathbf{e}_{z'}\ , \\
\frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{O'/O} &= {}^{\mathcal{I}}\mathbf{v}_{O'/O} = R\frac{{}^{\mathcal{I}}d}{dt}\mathbf{e}_{z'}\ , \\
{}^{\mathcal{I}}\mathbf{v}_{O'/O} &= R\left(\frac{{}^{\mathcal{R}}d}{dt}\mathbf{e}_{z'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{e}_{z'}\right)\ , \ \text{from transport eq. (19)} \\
&= R\omega(\mathbf{e}_z \times \mathbf{e}_{z'})\ , \ \because\ \frac{{}^{\mathcal{R}}d}{dt}\mathbf{e}_{z'} = \mathbf{0} \text{ and } {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\mathbf{e}_z \\
&= R\omega\left[\left(\cos\lambda\mathbf{e}_{y'} + \sin\lambda\mathbf{e}_{z'}\right) \times \mathbf{e}_{z'}\right]\ , \ \because\ \mathbf{e}_z = \cos\lambda\mathbf{e}_{y'} + \sin\lambda\mathbf{e}_{z'} \\
\therefore\ {}^{\mathcal{I}}\mathbf{v}_{O'/O} &= R\omega\cos\lambda\mathbf{e}_{x'}\ , \\
\frac{{}^{\mathcal{I}}d}{dt}\mathbf{v}_{O'/O} &= {}^{\mathcal{I}}\mathbf{a}_{O'/O} = R\omega\cos\lambda\frac{{}^{\mathcal{I}}d}{dt}\mathbf{e}_{x'}\ , \\
{}^{\mathcal{I}}\mathbf{a}_{O'/O} &= R\omega\cos\lambda\left(\frac{{}^{\mathcal{R}}d}{dt}\mathbf{e}_{x'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{e}_{x'}\right)\ , \ \text{from transport eq. (19)} \\
&= R\omega^2\cos\lambda(\mathbf{e}_z \times \mathbf{e}_{x'})\ , \ \because\ \frac{{}^{\mathcal{R}}d}{dt}\mathbf{e}_{x'} = \mathbf{0} \text{ and } {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\mathbf{e}_z \\
&= R\omega^2\cos\lambda\left[\left(\cos\lambda\mathbf{e}_{y'} + \sin\lambda\mathbf{e}_{z'}\right) \times \mathbf{e}_{x'}\right]\ , \ \because\ \mathbf{e}_z = \cos\lambda\mathbf{e}_{y'} + \sin\lambda\mathbf{e}_{z'} \\
\therefore\ {}^{\mathcal{I}}\mathbf{a}_{O'/O} &= R\omega^2\sin\lambda\cos\lambda\mathbf{e}_{y'} - R\omega^2\cos^2\lambda\mathbf{e}_{z'}\ .
\end{aligned}
$$

We therefore notice that ${}^{\mathcal{I}}\mathbf{a}_{O'/O}$ is also quadratic in $\omega$, which, as we have seen, is quite small. Therefore, $R\omega^2 \approx 0.03\,\mathrm{m/s^2}$, which, when multiplied by the trigonometric functions are generally even smaller. Therefore, to a first approximation, we can disregard the effects due to ${}^{\mathcal{I}}\mathbf{a}_{O'/O}$ as well. Hence, (24) effectively takes the form $m_P{}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{F}_P - 2m_P{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'}$; this implies, that to a first approximation, the dynamics of a particle as observed in the lab frame is fundamentally governed by the Coriolis acceleration term $2{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'}$. (The lab frame will become an inertial frame if the Coriolis acceleration term is also ignored.) The scalar equations of motion resulting from these approximations can now be obtained by substituting the terms ${}^{\mathcal{R}}\mathbf{a}_{P/O'} = \ddot{x}'\mathbf{e}_{x'} + \ddot{y}'\mathbf{e}_{y'} + \ddot{z}'\mathbf{e}_{z'}$, $F_P = -m_P g\mathbf{e}_{z'}$, and ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} = \left(\dot{z}'\omega\cos\lambda - \dot{y}'\omega\sin\lambda\right)\mathbf{e}_{x'} + \dot{x}'\omega\sin\lambda\mathbf{e}_{y'} - \dot{x}'\omega\cos\lambda\mathbf{e}_{z'}$ in $m_P{}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{F}_P - 2m_P{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'}$ and equating the coefficients of the corresponding unit vectors. We state the results as follows:

**Equations of Motion for a Free Falling Particle as Observed on Earth**

Consider an inertial frame $\mathcal{I}$ fixed at the origin $O$ of the Earth and a lab frame $\mathcal{R}$ with its origin $O'$ situated on Earth's surface. Then, to a first approximation, that is, to first order in Earth's angular speed, the motion of a point particle $P$ near the Earth's surface as described in the lab frame is governed by

$$m_P {}^{\mathcal{R}}\mathbf{a}_{P/O'} = \mathbf{F}_P - 2m_P {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'} \,, \tag{25}$$

where $2{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{R}}\mathbf{v}_{P/O'}$ is the Coriolis acceleration of the particle. The scalar equations of motion of a free falling particle, $P = (x', y', z')_{\mathcal{R}}$, according to a lab frame located at latitude $\lambda$ are:

$$\ddot{x}' = 2\omega \left(\dot{y}' \sin\lambda - \dot{z}' \cos\lambda\right), \tag{26}$$
$$\ddot{y}' = -2\omega \dot{x}' \sin\lambda, \tag{27}$$
$$\ddot{z}' = 2\omega \dot{x}' \cos\lambda - g. \tag{28}$$

The three scalar equations of motion, (26)–(28), are a set of coupled differential equations with $\omega$ appearing to first order in them (as we have ignored the $\omega^2$ terms in the EOM). The solutions to these equations in the context of a ball dropped from a tower can then be obtained as follows.

First, we differentiate (26) with respect to time, which results in $\dddot{x}' = 2\omega \left(\ddot{y}' \sin\lambda - \ddot{z}' \cos\lambda\right)$. Second, we substitute (27) and (28) in this latest expression to obtain

$$\frac{d^3x'}{dt^3} = -4\omega^2 \frac{dx'}{dt} + 2\omega g \cos\lambda. \tag{29}$$

This equation is a third-order uncoupled ordinary differential equation in $x'$. Note that it is not common to encounter third-order differential equations in mechanics (and in physics, in general). We can write (29) as

$$\frac{d^2\dot{x}'}{dt^2} = -4\omega^2 \dot{x}' + 2\omega g \cos\lambda,$$

which is simple harmonic in $\dot{x}'$. Therefore, a solution for $\dot{x}'$ is

$$\dot{x}' = \frac{g \cos\lambda}{2\omega} + A\cos(2\omega t), \tag{30}$$

where $A$ is a constant that depends on the initial conditions. Since we assume the ball (particle) to be dropped from rest from a tower of height $h$ directly above the origin of the lab frame, the initial conditions are,

$$(x'_0, y'_0, z'_0) = (0, 0, h); \ (\dot{x}'_0, \dot{y}'_0, \dot{z}'_0) = (0, 0, 0).$$

Therefore, at $t = 0$, $\dot{x}'_0 = 0$; substituting these in (30), we find $A = -g\cos\lambda/(2\omega)$. Hence,

$$\dot{x}' = \frac{g\cos\lambda}{2\omega}\left[1 - \cos(2\omega t)\right]. \tag{31}$$

Let us now assume that $\omega t \ll 1$, which implies that $t \ll 1/\omega$. Since $\omega \approx 7.3 \times 10^{-5}\ \mathrm{s}^{-1}$, $1/\omega \sim 4$ hours; so $t \ll 4$ hours. Therefore, for a ball that takes much less than 4 hours to fall,

$$\cos(2\omega t) = 1 - \frac{(2\omega t)^2}{2!} + \frac{(2\omega t)^4}{4!} - \cdots \approx 1 - 2\omega^2 t^2.$$

Substituting this approximation in (31) we obtain

$$\dot{x}' = g\omega\cos\lambda \cdot t^2. \tag{32}$$

Using the fact that at $t = 0$, $x'_0 = 0$,

$$\int_0^{x'} dx' = g\omega\cos\lambda \int_0^t t^2\, dt,$$

which simply yields,

$$\boxed{x' = \frac{1}{3} g\omega\cos\lambda \cdot t^3, \quad \omega t \ll 1 : \text{a first order effect in } \omega.}$$

This solution then allows us to determine the displacement of the ball in the $x'$-direction according to the lab frame at any given time $t$ (assuming that it takes much less than 4 hours for the fall). Note that the displacement in the $x'$-direction is first order in $\omega$. Except at the poles (where $\lambda = \pm\pi/2$), $\cos\lambda > 0$ regardless of whether $\lambda > 0$ (northern hemisphere) or $\lambda < 0$ (southern hemisphere), or $\lambda = 0$ (equator). Hence, except at the poles, the ball will land to the east ($x' > 0$) of the release point (recall the orientation of the lab frame in Figure 6). At the poles, $x' = 0$, and therefore, there is no deflection in the $X'$-direction. This is understandable, since at the poles, the ball is dropped along the Earth's rotational axis which does not distinguish any particular direction due to the fact that it is fixed in absolute space. In other words, a tower located at a pole has no movement relative to the inertial frame fixed at the center of the Earth. Another way to arrive at this conclusion is to refer to Figure 6 where at the location shown, $X'$-axis runs east-west. If this frame is transported to, say, the north pole, then the $Z'$-axis will point along the Earth's rotation axis (coinciding with the $Z$-axis of the inertial frame); but at this point, given the symmetry, there is no way to define a east-west direction using the $X'$-axis. Therefore, the displacement of the ball in the $X'$-direction must vanish.

In contrast, when the ball is dropped from a tower situated anywhere else, since the Earth (and therefore, the tower) is rotating eastward relative to the inertial frame fixed at the center of the Earth, a velocity component in the same direction will be imparted to the released ball. Since the ball is dropped from a height, it will have a greater eastward speed compared to that of the eastward speed of the base of the tower (relative to the inertial frame fixed at the center of the Earth). This difference then manifests as the net eastward deflection of the dropped ball according to the lab frame.

Now, let us substitute (32) in (27) to obtain $\ddot{y}' = -2g\omega^2 \sin\lambda \cos\lambda \cdot t^2$. Then, using the fact that at $t = 0$, $\dot{y}_0' = 0$,

$$\int_0^{\dot{y}'} d\dot{y}' = -2g\omega^2 \sin\lambda \cos\lambda \int_0^t t^2 \, dt \implies \dot{y}' = -\frac{2}{3} g\omega^2 \sin\lambda \cos\lambda \cdot t^3.$$

Integrating this expression again using the fact that at $t = 0$, $y_0' = 0$, we obtain

$$\boxed{y' = -\frac{1}{6} g\omega^2 \sin\lambda \cos\lambda \cdot t^4, \quad \omega t \ll 1 : \text{a second order effect in } \omega.}$$

Therefore, it is evident that the displacement of the ball in the $Y'$-direction is a second order effect in $\omega$. Again, at the poles ($\lambda = \pm\pi/2$), $y' = 0$, and therefore, there is no deflection of the ball in the $Y'$-direction since a tower located at a pole has no movement relative to the inertial frame fixed at the center of the Earth. Another way to arrive at this conclusion is to refer to Figure 6 where at the location shown, $Y'$-axis runs north-south. If this frame is transported to, say, the north pole, then the $Z'$-axis will point along the Earth's rotation axis (coinciding with the $Z$-axis of the inertial frame); but at this point, given the symmetry, there is no way to define a north-south direction using $Y'$-axis. Therefore, the displacement of the ball in the $Y'$-direction must vanish.

Similarly, unlike the eastward deflection at the equator, there is no deflection of the ball in the $Y'$-direction if dropped at the equator. In the lab frame set up at the equator, its $Y'$-axis runs north-south. Therefore, if the ball is dropped from a tower there, given the symmetry of the Earth as seen at the equator, how can the ball decide whether to veer north or south? This symmetry, then, determines the lack of displacement of the ball in the $Y'$-direction when dropped from a tower at the equator. In other locations, since our lab frame is oriented such that positive $y'$ implies north (see Figure 6), we see that a ball dropped in the northern hemisphere ($\lambda > 0$) will be deflected south ($y' < 0$); in contrast, a ball dropped in the southern hemisphere ($\lambda < 0$) will be deflected north ($y' > 0$). However, these non-zero deflections are quite small since the effect is second order in $\omega$.

Finally, let us substitute (32) in (28) to obtain $\ddot{z}' = g(2\omega^2 \cos^2 \lambda \cdot t^2 - 1)$. Then, using the fact that at $t = 0$, $\dot{z}'_0 = 0$,

$$\int_0^{\dot{z}'} d\dot{z}' = g \int_0^t (2\omega^2 \cos^2 \lambda \cdot t^2 - 1)\, dt \implies \dot{z}' = g\left(\frac{2}{3}\omega^2 \cos^2 \lambda \cdot t^3 - t\right).$$

Integrating this expression again using the fact that at $t = 0$, $z'_0 = h$, we obtain

$$\boxed{z' = h + g\left(\frac{1}{6}\omega^2 \cos^2 \lambda \cdot t^4 - \frac{1}{2}t^2\right), \quad \omega t \ll 1.}$$

Note that a second order effect in $\omega$ is also present for the displacement in $z'$. Let us summarize the above solutions as follows:

(c) Motion of a ball dropped from a tower as observed on Earth

Consider a ball dropped from rest at height $h$ from a tower located at latitude $\lambda$. Suppose the lab frame $\mathcal{R} \equiv \{O', \mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}\}$ is oriented such that $\mathbf{e}_{x'}$ points east, $\mathbf{e}_{y'}$ points north, and $\mathbf{e}_{z'}$ points directly upward. Then, given the initial conditions $(x'_0, y'_0, z'_0) = (0, 0, h)$ and $(\dot{x}'_0, \dot{y}'_0, \dot{z}'_0) = (0, 0, 0)$, the position $P = (x', y', z')_{\mathcal{R}}$ of the ball according to the lab frame at any time $t$, such that $\omega t \ll 1$, where $\omega$ is the angular speed of the Earth, is given by:

$$x' = \frac{1}{3} g\omega \cos \lambda \cdot t^3, \tag{33}$$

$$y' = -\frac{1}{6} g\omega^2 \sin \lambda \cos \lambda \cdot t^4, \tag{34}$$

$$z' = h + g\left(\frac{1}{6}\omega^2 \cos^2 \lambda \cdot t^4 - \frac{1}{2}t^2\right). \tag{35}$$

The first order in $\omega$ (prominent) deflection, $x'$, is to the east except at the poles ($\lambda = \pm\pi/2$); the eastward deflection is highest at the equator ($\lambda = 0$); the second order in $\omega$ (minor) deflection, $y'$, is to the north in the southern hemisphere ($\lambda < 0$) and is to the south in the northern hemisphere ($\lambda > 0$), except at the poles and at the equator ($\lambda = 0$).

If we ignore the second order effect in $\omega$ in $z'$, then $z' \approx h - \frac{1}{2}gt^2$, which is a most familiar expression. The ball then reaches the ground ($z' = 0$) at time $t = T$ so that

$$0 = h - \frac{1}{2}gT^2 \implies T = \sqrt{\frac{2h}{g}}.$$

The total $x'$ and $y'$ displacements of the ball during this time are then,

$$\boxed{x'|_{\text{east}} = \frac{1}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}}, \quad y'|_{\text{north (s.h. : }\lambda<0)/\text{south (n.h. : }\lambda>0)} = -\frac{2}{3}\cdot\frac{h^2}{g}\cdot\omega^2\sin\lambda\cos\lambda.} \tag{36}$$

The eastward deflection of the ball is maximum at the equator ($\lambda = 0$), while the north/south deflection is maximum halfway to the poles ($\lambda = \pm\pi/4$; n.h. and s.h. stand for the northern and the southern hemisphere, respectively). In contrast, as described earlier, there is no north/south deflection at the equator; and there is no deflection of any kind at the poles ($\lambda = \pm\pi/2$). Figure 8 shows the deflections of a ball dropped from the top of the Eiffel Tower, which is about 312 m high. Such a ball will land about 8 cm to the east from where it was dropped; its southward deflection is about 17 micrometers, and hence, is almost null. The eastward deflection of the ball is still small, and may appear to an observer at the base of the tower to land almost directly below where it was released. This example then shows the difficulty of discerning the rotation of the Earth by observing falling objects, which practically fall from much smaller heights resulting in even smaller deflections.

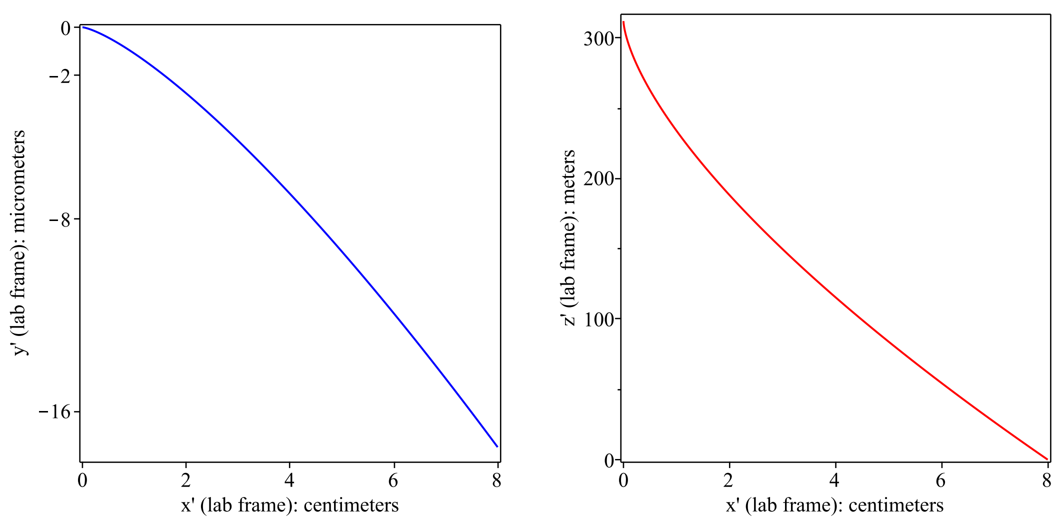


Figure 8: Motion of a ball dropped from the top of the Eiffel Tower: height $h \approx 312$ m, latitude $\lambda \approx 48.86°$ (0.85 rad) N. It takes the ball close to 8 s to fall to the ground; $g = 9.81$ m/s$^2$. Left (blue): Eastward ($x' > 0$) deflection of the ball against its southward ($y' < 0$) deflection; the ball lands just short of 8 cm to the east and about 17 $\mu$m to the south from where it was released. Right (red): Eastward ($x' > 0$) deflection of the ball against the height ($z' \geq 0$).

**(d) Motion of a Ball Thrown Vertically Upward from a Point on Earth**

We now consider a ball thrown vertically upward from the origin $O'$ of the lab frame located at latitude $\lambda$. The equations of motion (26)–(28) still apply but now with the initial conditions

$$(x'_0, y'_0, z'_0) = (0,0,0); \ (\dot{x}'_0, \dot{y}'_0, \dot{z}'_0) = (0,0,v_0),$$

where $v_0$ is the initial vertical speed of the ball as measured in the lab frame. Since the initial conditions in the $X'$- and $Y'$-directions are the same as before, it follows (as before) that, for $\omega t \ll 1$ (that is, for motions with duration much less than 4 hours),

$$\dot{x}' = g\omega \cos\lambda \cdot t^2, \ \ \dot{y}' = -\frac{2}{3} g\omega^2 \sin\lambda \cos\lambda \cdot t^3.$$

Since $\dot{y}'$ is second order in $\omega$, we can effectively ignore this term. Similarly, any term having the combination $\omega\dot{x}'$ is also second order in $\omega$; such terms, therefore, can also be effectively ignored. The equations of motion (26)–(28) then reduce to

$$\ddot{x}' = -2\omega \dot{z}' \cos\lambda, \tag{37}$$
$$\ddot{y}' = 0, \tag{38}$$
$$\ddot{z}' = -g. \tag{39}$$

Given the initial conditions, it is evident from (38), that $y' = 0 \ \forall t$. From (39), using the initial conditions, it also follows from integration that

$$\dot{z}' = v_0 - gt, \quad z' = v_0 t - \frac{1}{2} g t^2,$$

which are familiar expressions. The ball lands on the ground when $z' = 0$ at $t = T > 0$. It therefore follows from $0 = v_0 T - \frac{1}{2} g T^2 = (v_0 - \frac{1}{2} g T)T$, that the ball is in flight for a duration of $T = 2v_0/g$. Hence, it take the ball $t = \tau = v_0/g$ to reach the maximum height $h$ of its trajectory. Therefore, $h = v_0\tau - \frac{1}{2} g\tau^2 = v_0^2/(2g)$; solving this for $v_0$, we obtain $v_0 = \sqrt{2gh}$. Thus, $T = 2\sqrt{2h/g}$. Now, substituting $\dot{z}' = v_0 - gt$ in (37), we obtain $\ddot{x}' = -2\omega \cos\lambda (v_0 - gt)$. Using the initial conditions, this expression can be directly integrated to yield

$$\dot{x}' = -2\omega \cos\lambda \left( v_0 t - \frac{1}{2} g t^2 \right), \quad x' = -\omega \cos\lambda \left( v_0 t^2 - \frac{1}{3} g t^3 \right).$$

Note that the solution for $x'$ is a combination of two competing terms: one, quadratic, and the other, cubic, in time. To understand the net effect of these two competing terms note that

$$\left(v_0 t^2 - \frac{1}{3} g t^3\right) = 2\int_0^t \left(v_0 t - \frac{1}{2} g t^2\right) dt = 2\int_0^t z' \cdot dt \geq 0, \ \because\ z' \geq 0 \ \ \forall\ t \geq 0.$$

Therefore, the net effect of the two competing time terms in the solution for $x'$ is either null (when $t = 0$) or positive (when $t > 0$). Therefore, $x' < 0$ for $t > 0$. Since $\cos\lambda > 0$ at any point on Earth (except at the poles), and that $\mathbf{e}_{x'}$ is oriented to the east, the negative $x'$ values imply that, other than at poles, the ball will land to the west of where it was launched. There is no deflection of the ball at the poles. We summarize the results below.

(d) Motion of a ball thrown vertically upward from a point on Earth

A lab frame $\mathcal{R} \equiv \{O', \mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}\}$ at latitude $\lambda$ is oriented such that $\mathbf{e}_{x'}$ points east, $\mathbf{e}_{y'}$ points north, and $\mathbf{e}_{z'}$ points directly upward. A ball is thrown vertically upward at $O'$ with initial velocity $\mathbf{v}_0 = v_0 \mathbf{e}_{z'}$. Let $\omega$ be the angular speed of the Earth, and let the second order effects of $\omega$ be ignored from the equations of motion. Then, given the initial conditions $(x'_0, y'_0, z'_0) = (0, 0, 0)$ and $(\dot{x}'_0, \dot{y}'_0, \dot{z}'_0) = (0, 0, v_0)$, the position $P = (x', y', z')_{\mathcal{R}}$ of the ball according to the lab frame at any time $t$, such that $\omega t \ll 1$, is given by:

$$x' = -\omega\cos\lambda\left(v_0 t^2 - \frac{1}{3} g t^3\right), \tag{40}$$

$$y' = 0, \tag{41}$$

$$z' = v_0 t - \frac{1}{2} g t^2. \tag{42}$$

The first order in $\omega$ deflection, $x'$, is to the west except at the poles ($\lambda = \pm\pi/2$) where it is null; the westward deflection is highest at the equator ($\lambda = 0$).

Since the ball takes a time $T = 2\sqrt{2h/g}$ to fall back on the ground, the total displacement in the $x'$-direction is given by $x'(T) = -\omega\cos\lambda\left(v_0 T^2 - \frac{1}{3} g T^3\right)$, which, along with $v_0 = \sqrt{2gh}$, evaluates to

$$\boxed{x'|_{\text{west}} = -\frac{4}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}}.} \tag{43}$$

Thus, except at the poles, when a ball is dropped from rest, it lands to the east of the release point; in contrast, when a ball is thrown vertically upward, it lands to the west of the launch point. Figure 9 shows the westward deflection against the height of a ball launched at an initial vertical speed of $v_0 = 78$ m/s. Such an initial speed will allow the ball to reach the top of the

Eiffel Tower (height, $h \approx 312$ m). The ball lands about 32 cm (about a foot) to the west of the launch point. With this example, we again realize the difficulty of detecting the rotation of the Earth about its axis by simply throwing a ball up in the air, where typical throws would not carry a projectile as high as the Eiffel Tower, and therefore, would result in much smaller deflections.

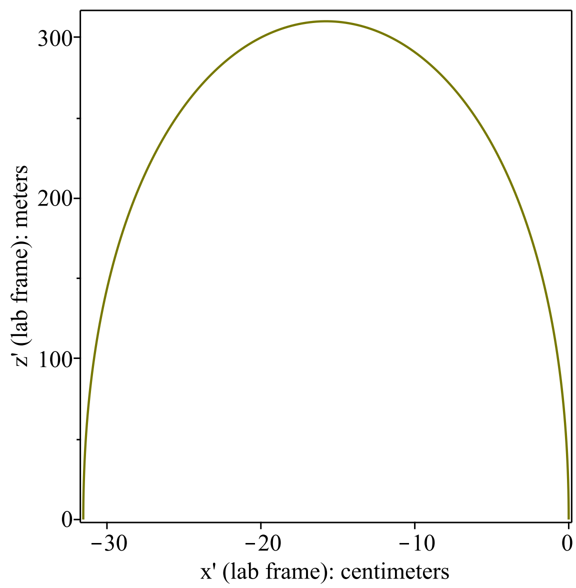


Figure 9: The westward ($x' < 0$) deflection against the height ($z' \geq 0$) of a ball thrown vertically upward at an initial speed $v_0 = 78$ m/s at latitude $\lambda \approx 48.86°$ (0.85 rad) N. This initial speed will allow the ball to reach the top of the Eiffel Tower (height, $h \approx 312$ m). It takes the ball about 16 s to fall back to the ground; $g = 9.81$ m/s$^2$. The ball lands about 32 cm (about a foot) to the west of the launch point.

Given the discussion above, we can therefore appreciate the effort it took in human history to realize that the Earth, in fact, is in motion. The problem of motion, as the reader may have glimpsed from this essay, is non-trivial, and it played a profound role in the development of mechanics, and science in general.

Historically, Newton realized that due to Earth's rotation about its axis, a ball dropped from a tower must land to the east of the release point contrary to earlier beliefs that it should land to the west. He estimated this eastward deflection at the equator to be $\omega\sqrt{2h^3/g}$, which is a factor of 3/2 greater than the effect stated in (36). The reason for this discrepancy is due to the fact that Newton did not account for the Coriolis acceleration which has a component directed toward the west, which in turn reduces the net eastward deflection. In his correspondence with Newton, Robert Hooke (1635-1703) made many important observations and clarifying comments regarding the nature of motion and the forces acting on bodies. Hooke realized, correctly, that in addition to the eastward deflection, a falling object should also deflect south in the northern hemisphere. However, Hooke thought that this southward deflection is the primary effect, when in fact, as stated in (36), it is a second order effect in $\omega$, and therefore, is quite small. Regarding the Coriolis acceleration, its mathematical description was provided by Gaspard-Gustave Coriolis (1792-1843) in 1835. However, its effect on falling bodies as observed on Earth have been foreseen by Giovanni Battista Riccioli (1598-1671) and Claude François Milliet Dechales (1621-78) close to two centuries before Coriolis. Interestingly, though, they both rejected the rotation of the Earth since they were unable to observe the deflections due to Coriolis acceleration. This historical snippet is shared with the reader to illustrate that the problem of falling bodies posed a significant challenge to the pioneers of mechanics. The problem required careful observation and the development of new physical and mathematical insights to understand.

**Further Reading**

The history of astronomy and mechanics, and the cultural inputs to its development and resulting implications is a vast subject in itself. Some literature that the readers may find useful within the context of the present essay are:


- V. I. Arnol'd, *Huygens & Barrow, Newton & Hooke*, Eric Primrose (translator), Birkhäuser, Basel, 1990.
- I. Bernard Cohen, *The Birth of a New Physics*, W. W. Norton, New York, 1985.
- Galileo Galilei, *Dialogue Concerning the Two Chief World Systems*, Stillman Drake (translator), Stephen J. Gould (series editor), The Modern Library, New York, 2001.
- Owen Gingerich, *The Eye of Heaven: Ptolemy, Copernicus, Kepler*, American Institute of Physics, New York, 1993.
- John North, *Cosmos*, The University of Chicago Press, Chicago, 2008.
- Károly Simonyi, *A Cultural History of Physics*, David Kramer (translator), CRC Press, Boca Raton, 2012.


For the systematic approach to kinematics and dynamics, along with the notation we have adopted, the reader is referred to:


- N. Jeremy Kasdin and Derek A. Paly, *Engineering Dynamics: a comprehensive approach*, Princeton University Press, Princeton, 2011.


♣