Summoned from the Void: the Casimir Effect and the Quantum Vacuum

Rasil Warnakulasooriya

> [L]et us be glad there existed and exist people like [Wolfgang] Pauli, who rate a profound and beautiful theory higher than any practical application .... During an excursion by boat on the Neckar I explained to him my results on the Van der Waals forces and their relation to field fluctuations in empty space. He began by bluntly telling me it was all nonsense, but was obviously amused when I did not give in. Finally, after I had countered all his arguments, he agreed, ... It was the last time he did not call me "Herr Direktor." –Hendrik B. G. Casimir, *Haphazard Reality: Half a Century of Science* (Harper & Row, 1983).

1 Introduction

The notion of "nothingness" has been contemplated by ancient philosophers both in terms of materialistic- and spiritual-senses[1]. As attributed to Aristotle, "nature abhors a vacuum." In principle, a material void can be created in a container by removing all the material (e.g., air) particles in it. The barometer of Evangelista Torricelli, constructed in 1644, is a practical example of an apparatus where the space above the mercury column of the inverted tube can be considered to be void of any air. A more dramatic example is provided by what are known as the Magdeburg hemispheres created by Otto von Guericke in 1654. Once the two hemispheres were combined (without any fastening mechanisms) and the air inside the resulting sphere removed, two teams of eight draft horses were not able to separate them (see Figure 1).

However, it was realized in the nineteenth century that a container void of material particles is filled with thermal radiation, the study of which lead to quantum physics at the turn of the twentieth century. In quantum theory we have to consider the walls of the container as being in equilibrium with the electromagnetic thermal radiation within its space via the exchange of discrete packets of energy (quanta). This thermal radiation is not monochromatic but is distributed across the frequency spectrum of the electromagnetic field. The exact nature of the frequency spectrum, that is, for example, which frequency corresponds to the highest intensity of the thermal radiation, depends on the temperature of the container wall: higher the temperature, higher the frequency contributing to highest intensity of the thermal radiation. If the container (shielded or isolated from the rest of the universe) can be cooled to absolute zero temperature (that is, zero degree kelvin or 0 K), then, according to Planck's radiation law, this thermal radiation inside the container will vanish. Lacking both material particles and thermal radiation inside, one then wonders whether a true void has been achieved inside the container.

Alas, Aristotle's view that "nature abhors a vacuum" seems to be substantive after all. Even when the container is void of both material particles and thermal radiation, there still is something

[1] For a history of ideas regarding the vacuum, see *Much Ado about Nothing: Theories of Space and Vacuum from the Middle Ages to the Scientific Revolution* by Edward Grant (Cambridge University Press, 2008).

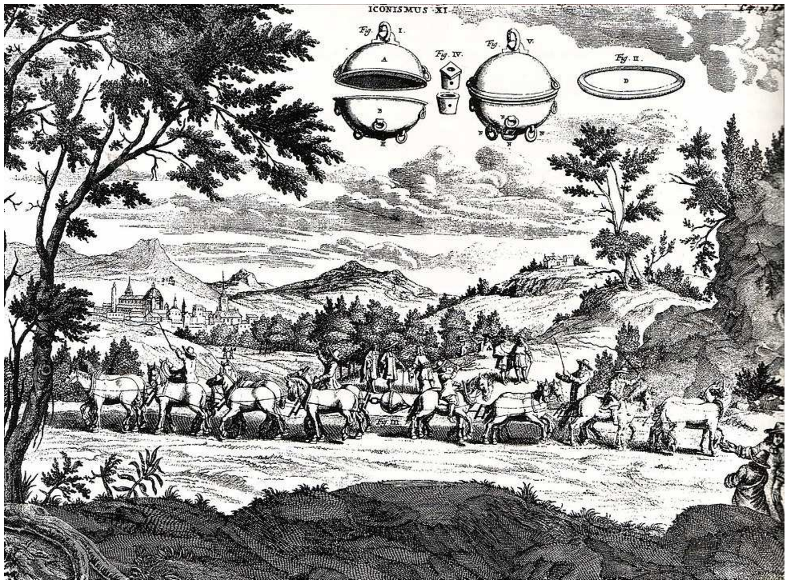

Figure 1: The experiment of Otto von Guericke performed in 1654. Two hemispheres were combined (without any fastening mechanisms) and the air inside the resulting sphere was removed. Two teams of eight draft horses were not able to separate the hemispheres. Engraving by Gaspar Schott | public domain.

remaining inside the container. This "something" is known as vacuum fluctuations. In other words, the void inside a container, lacking in material particles and thermal radiation, is not an absolute void. According to Heisenberg's uncertainty principle, virtual electromagnetic quanta (i.e., virtual photons) can exist within the void. They are virtual since they cannot be detected by instruments as their existence is so brief in violation of the energy conservation. One mental model the physicists use in this case is to imagine that the virtual photons appear from the void and disappear into the void. Thus, the vacuum, far from being a space of nothingness, is filled with virtual particle dynamics. In what follows, the term "vacuum" refers to a space void of material particles that may or may not be at a temperature of absolute zero.

Though the virtual particles themselves are unobservable, their existence do seem to result in effects that are measurable. In 1948, Hendrik Casimir theoretically explored[2] the effect of the vacuum on two identical, rectangular, massless, neutral, perfectly conducting (perfectly reflecting) parallel plates, the entire system being at absolute zero temperature. His startling conclusion was that the two plates must attract each other with the force of attraction varying as the inverse fourth-power of the distance between the two plates. Since the plates were massless, no gravitational effect between them can arise; similarly, since the plates are neutral, electrostatic force between them can also be ruled out. Since only the virtual electromagnetic quanta or the field

[2]See *On the attraction between two perfectly conducting plates*, H. B. G. Casimir, Proceedings of the Royal Netherlands Academy of Arts and Sciences (Proc. K. Ned. Akad. Wet.), 51 (1948), 793-795.

is accounted for, it seems that the attraction between the two plates must be attributed to the vacuum. Most interestingly, this effect, which has come to be known as the Casimir Effect, has been experimentally verified.[3] Nothingness, indeed, materializes in concrete measurable ways. Both interesting physics and mathematics are embedded within the Casimir Effect. The present essay aims to take a leisurely stroll through this garden.

2 Casimir Effect at Absolute Zero Temperature

2.1 The set up

Two identical, massless, neutral, perfectly conducting (perfectly reflecting) rectangular plates of negligible thickness are placed in the vacuum at absolute zero temperature ($T = 0$ K) such that their surfaces are parallel to each other and are a distance r apart. The coordinates are chosen so that the surface of each plate is parallel to the y-z plane. Each plate has length l_y in the y-direction and length l_z in the z-direction; the separation r then falls along the x-direction (see Figure 2). The separation between the plates is much smaller than the dimensions of a plate: that is, $r << l_y,\ l_z$.

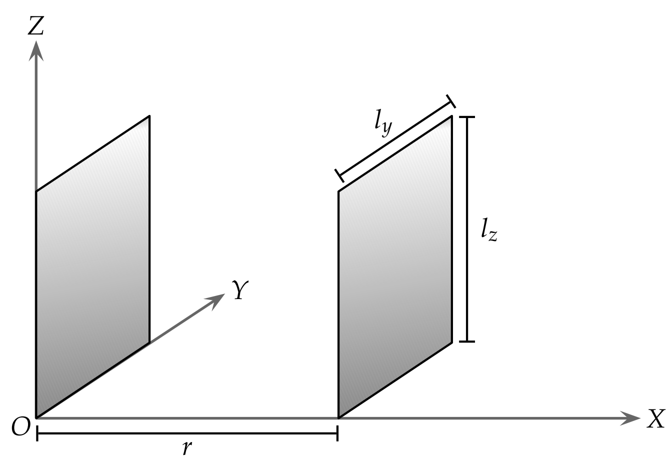

Figure 2: Two identical, rectangular, massless, neutral, perfectly conducting parallel plates placed a distance r apart along the x-direction in the vacuum. The area A of a plate is $l_y l_z$. Though exaggerated in the figure, $r << l_y,\ l_z$. | Drawing by AD.

[3]For a review of the theory, experiments, applications, and outstanding questions, see *Casimir forces: Still surprising after 60 years* by Steve K. Lamoreaux, Physics Today, February, 2007, 40-45.

According to Heisenberg's uncertainty principle, there are electromagnetic field fluctuations even if there are only virtual photons within the vacuum space. These electromagnetic field fluctuations that are related to virtual photons are to be understood as vibrations of the electric and magnetic fields about their zero average values. Further, these fluctuations can be divided into two parts: fluctuations that take place within the space bounded by the plates (we have to assume imaginary boundaries in the y- and z-directions), and fluctuations that take place outside of it. Simply put, the Casimir effect arises due to the vacuum energy density difference between the space bounded by the plates and the space outside of it.

To see the proverbial forest before getting lost among the trees, the basic computational approach may be laid out as follows: We first compute the electromagnetic field energy of the space (vacuum) bounded by the plates. This vacuum energy is technically infinite when summed over all the (angular) frequency modes of the field. Encountering this infinity may immediately dash our hopes of obtaining sensible answers. However, at this point, we can introduce the notion of regularization whereby the perfectly conducting plates become transparent to electromagnetic waves with frequencies beyond a certain cut-off frequency. For electromagnetic waves at or below the cut-off frequency the perfectly conducting plates are totally opaque, and therefore, such waves are fully contained within the vacuum space bounded by the plates. (This, hence, explains the demand for perfectly conducting plates in observing the Casimir effect.) This approach then leads to the energy of the vacuum bounded by the plates, which we can use to derive the force exerted on the plates by the vacuum within. This force takes the form $\sum f(x)$, where $f(x)$ is some function of a variable x. The force exerted on the plates by the vacuum outside of the plates can then be argued to take the form $\int f(x)dx$. The net force on the plates is then given by the difference of these two expressions, $\sum f(x) - \int f(x)dx$. Though $\sum f(x)$ and $\int f(x)dx$ are each separately infinite, their difference is finite. (This is a concrete example of the oft quoted saying in popular accounts of quantum field theory that "infinites cancel each other to yield finite answers.") Here, interesting mathematics arises, namely, the Euler-Maclaurin summation. In fact, the evaluation of $\sum f(x) - \int f(x)dx$ via the Euler-Maclaurin summation is equivalent to directly considering the divergent series $\sum f(x)$ (via analytic continuation) with a unit regularization according to Ramanujan summation. It is therefore interesting that the finite values assigned to divergent series by Ramanujan may have more to do with reality than how divergent series were originally viewed by mathematicians.[4] We now embark on the calculation proper.

2.2 Angular frequency of a wave bounded by the plates

We now set out to obtain an expression for the angular frequency modes of the electromagnetic waves in the vacuum bounded by the plates. Let $\mathbf{x} = (x, y, z)$ be the position vector of a point in

[4] For example, in his Preface to G. H. Hardy's text *Divergent Series* (Oxford, 1949), J. E. Littlewood says the following: "Abel wrote in 1828: 'Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.'" As Hardy says in his text, "it is broadly true to say that mathematicians before Cauchy asked not 'How shall we *define* $1 - 1 + 1 - \ldots$?' but 'What *is* $1 - 1 + 1 - \ldots$?', and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal."

space and $\boldsymbol{\kappa} = (k_x, k_y, k_z)$ be the wave vector of an electromagnetic wave. Our immediate goal is to obtain relations that express the components of the wave vector $\boldsymbol{\kappa}$ so that an expression for angular frequency modes of the electromagnetic waves can be derived.

Since these waves fully reflect between the plates along the x-direction, a non-negative integer number of half-wavelengths must fit within the plates in the x-direction. This is the case for any standing wave (for example, a guitar string tied between the ends). Therefore,

$$r = m_x \frac{\lambda_x}{2}, \quad m_x \in \mathbb{N},$$

where λ_x is the wavelength of a standing wave between the plates in the x-direction in the vibrational mode m_x. Since the wave number k_x is equivalent to $2\pi/\lambda_x$, it is the case that,

$$k_x = \frac{m_x \pi}{r} \geq 0, \quad r \neq 0.$$

Let us now consider a wave in the y-direction. The electromagnetic wave solutions take on the form

$$\Phi(\mathbf{x}, t) = \mathcal{A} \exp[-i(\omega t - \boldsymbol{\kappa} \cdot \mathbf{x})],$$

where $\mathcal{A}$ is the amplitude of the wave, ω is its angular frequency, t is the time, and $i = \sqrt{-1}$. Since there are no hard boundaries along the y-direction, we can impose the periodic boundary condition that the wave solution takes on the same value at $y = 0$ (where we take one edge of both plates to be located; see Figure 2) and at $y = l_y$ at any given instant t. Thus, imposing $\Phi(y = 0, t) = \Phi(y = l_y, t)$ yields

$$\mathcal{A} \exp[-i(\omega_y t - k_y(0))] = \mathcal{A} \exp[-i(\omega_y t - k_y(l_y))] \implies \exp(i k_y l_y) = 1.$$

Since $\exp(i k_y l_y) = \cos(k_y l_y) + i \sin(k_y l_y)$, this implies that the wave along the y-direction must satisfy,

$$\cos(k_y l_y) + i \sin(k_y l_y) = 1 \implies k_y l_y = 2\pi m_y \implies k_y = \frac{2\pi m_y}{l_y} \geq 0, \quad m_y \in \mathbb{N}.$$

It is clear that a similar expression holds for wave vectors in the z-direction, so that $k_z = 2\pi m_z / l_z \geq 0, \quad m_z \in \mathbb{N}$. Thus the square magnitude (κ^2) of the wave vector for a given electromagnetic wave is:

$$\boldsymbol{\kappa} \cdot \boldsymbol{\kappa} = \kappa^2 = k_x^2 + k_y^2 + k_z^2 = \frac{m^2\pi^2}{r^2} + k^2, \text{ where } m = m_x, \; k^2 = \mathbf{k} \cdot \mathbf{k} = k_y^2 + k_z^2.$$

The wave vector is therefore dependent on two variables $(m, \mathbf{k})$. Taking the speed of light in the vacuum to be c, the angular frequency of a wave bounded in the space between the plates is then

$$\boxed{\omega_m(\mathbf{k}) = c\kappa = c\left(\frac{m^2\pi^2}{r^2} + k^2\right)^{1/2}, \; \omega_m(\mathbf{k}) \geq 0, \; m \in \mathbb{N}, \; k \geq 0, \; r \neq 0.} \tag{1}$$

The angular frequencies of the vibrations are defined as non-negative so that the vacuum energy considered below is non-negative. The particular form of (1) highlights only the dependence on the non-negative integer $m(= m_x)$ along with $k^2(= k_y^2 + k_z^2)$; the latter appears without having its form elaborated in terms of the non-negative integers m_y and m_z. The reason for doing so stems from the fact that the distance (r) between the parallel plates is much smaller than the dimensions (l_y, l_z) of the plates: that is, $r << l_y, \; l_z$, and therefore, vibrations are confined to a much smaller space in the x-direction in contrast to the y- and z-directions. Thus, we can take the vibrational modes to be discrete in the x-direction and continuous in the y- and z-directions. As such, in what follows, we will take m as discrete with k_y and k_z continuous. We also note that the expression in (1) is general and it applies to any wave whether it is in the vacuum or not. (The reader may have noticed that none of the expressions stated above explicitly depended on the fact that the plates have been placed in a vacuum.) Additionally, we have not yet invoked any concepts that are inherently quantum mechanical in nature (that is, Planck's constant h is absent so far). We now enter the vacuum energy, which is specific to the problem at hand.

2.3 The vacuum energy in the space bounded by the plates

According to the quantum theory, at absolute zero temperature, a vacuum fluctuation with angular frequency $\omega_m(\mathbf{k})$ has energy

$$\boxed{\epsilon_0 = \frac{1}{2}\hbar\omega_m(\mathbf{k}), \; \text{vacuum or zero-point energy}} \tag{2}$$

where $\hbar = h/(2\pi)$ and h is the Planck's constant. Since we take these fluctuations to be electromagnetic in nature, we have to consider two vibrational (or polarization) modes for a given $\omega_m(\mathbf{k})$: that is, two transverse directions with respect to a given direction for a given $\omega_m(\mathbf{k})$ except when $m(= m_x)$ or m_y or m_z is zero, in which case only a single mode is available to a wave. As described earlier, since we will consider only m to be discrete (with k_y and k_z being continuous), the two different modes of vibrations need to be tracked only for m: if $m = 0$, only a single vibrational mode will be available to the wave (hence non-degenerate); if $m \neq 0$, two vibrational modes of equal energy will be available to the wave (hence degenerate). Thus, the total energy of a wave for a given angular frequency $\omega_m(\mathbf{k})$ is

$$E_0 = \epsilon_0|_{m=0} + 2\epsilon_0|_{m\neq 0} = \frac{1}{2}\hbar\omega_m(\mathbf{k})|_{m=0} + \hbar\omega_m(\mathbf{k})|_{m\neq 0}.$$

To obtain the total vacuum energy in the space bounded by the plates ($\mathcal{E}_{0_{\text{in}}}$), we need to sum E_0 over all m and $\mathbf{k}$, thereby obtaining

$$\boxed{\mathcal{E}_{0_{\text{in}}} = \sum_{[m],\mathbf{k}} E_0 = \sum_{[m],\mathbf{k}} \hbar\omega_m(\mathbf{k}) \to \infty, \quad m \in \mathbb{N}, \quad k \geq 0. \text{ total vacuum or zero-point energy}} \tag{3}$$

Per the discussion above, the symbol $[m]$ in the above sum signifies that $\hbar\omega_m(\mathbf{k})$ is to be multiplied by a $1/2$ factor when $m = 0$ with the summation over m being discrete and the summation over $\mathbf{k}$ being continuous. As mentioned earlier, since we demand energies to be non-negative[5], $\omega_m(\mathbf{k}) \geq 0$; as a result, the sum expressed in (3) yields infinity for vacuum energy for the space bounded between the plates if we let the angular frequencies to have values without any limit. We therefore seek to limit the allowed frequencies via the method of regularization.

2.4 Regularization of the vacuum energy in the space bounded by the plates

To tame the infinity in (3), let us propose a regularization term ($\mathcal{R}$) such that

$$\mathcal{R}(\omega_m(\mathbf{k})/\omega_c) = \exp[-\omega_m(\mathbf{k})/\omega_c], \quad \omega_c = \text{cut-off angular frequency.} \tag{4}$$

It is clear, therefore, that

$$\frac{\omega_m(\mathbf{k})}{\omega_c} \to \infty \implies \mathcal{R}(\omega_m(\mathbf{k})/\omega_c) \to 0; \quad \frac{\omega_m(\mathbf{k})}{\omega_c} = 0 \implies \mathcal{R}(\omega_m(\mathbf{k})/\omega_c) = 1.$$

As such, for $\omega_m(\mathbf{k}) > \omega_c$, as $\omega_m(\mathbf{k})$ increases, $\mathcal{R}(\omega_m(\mathbf{k})/\omega_c)$ decreases. We can now augment (3) by the regularization factor such that

$$\boxed{\mathcal{E}_{0_{\text{in}}} = \sum_{[m],\mathbf{k}} \hbar\omega_m(\mathbf{k})\mathcal{R}(\omega_m(\mathbf{k})/\omega_c). \text{ regularized total vacuum or zero-point energy}} \tag{5}$$

Since ω_c is an arbitrarily imposed cut-off frequency, we can posit that it must not feature in the observables of interest such as the force between the plates. We now set out to compute (5). For this, let us first focus on the summation term itself. The symbol $\sum_{m,\mathbf{k}}$ signifies that we need to

[5] Negative energies appear in relativistic quantum mechanics, which are interpreted as (real) antiparticles: entities with opposite charge but same mass as particles. In the case of the vacuum at absolute zero temperature considered here, no actual particles or antiparticles exist. Also, in contrast to relativistic quantum mechanics, all energies are non-negative in quantum field theory.

sum over both m and $\mathbf{k}$ with the summation over m being discrete and the summation over $\mathbf{k}$ being continuous as described earlier. Noting that the summation over $\mathbf{k}$ is actually two sums over $m_y = l_y k_y/(2\pi)$ and $m_z = l_z k_z/(2\pi)$, turning the summation over $\mathbf{k}$ to a continuous integral amounts to,

$$\sum_{\mathbf{k}} \to \int (dm_y)(dm_z) = \frac{l_y l_z}{(2\pi)^2} \int (dk_y)(dk_z) = \frac{A}{(2\pi)^2} \int d^2k,$$

where $A = l_y l_z$ is the inner area of a plate.[6] The summation over m ranges from 0 to ∞. Hence,

$$\sum_{[m],\mathbf{k}} = \sum_{[m]} \sum_{\mathbf{k}} = \frac{A}{(2\pi)^2} \sum_{m=[0]}^{\infty} \int d^2k. \tag{6}$$

Note that the symbol $m = [0]$ denotes the need to insert a $1/2$ factor in the sum when $m = 0$. Therefore,

$$\mathcal{E}_{0_{\text{in}}} = \frac{A\hbar}{(2\pi)^2} \sum_{m=[0]}^{\infty} \int \omega_m(\mathbf{k}) \mathcal{R}(\omega_m(\mathbf{k})/\omega_c)\ d^2k.$$

Our goal now is to evaluate this expression, which would yield the total vacuum (or zero-point) energy for the space bounded by the plates. We first recall that the integration over d^2k refers to integration over the two-dimensional (y-z) planes in k-space. Thus, considering a small slice of the two-dimensional k-space with radius k bounded by an angle $d\phi$ and thickness dk,

$$d^2k = (k\ d\phi)\ dk.$$

Here, k ranges from 0 to ∞ and ϕ ranges from 0 to 2π. Thus

$$\mathcal{E}_{0_{\text{in}}} = \frac{A\hbar}{(2\pi)^2} \sum_{m=[0]}^{\infty} \int_0^{2\pi} d\phi \int_0^{\infty} \omega_m(\mathbf{k}) \mathcal{R}(\omega_m(\mathbf{k})/\omega_c)\ k\ dk = \frac{A\hbar}{2\pi} \sum_{m=[0]}^{\infty} \int_0^{\infty} \omega_m(\mathbf{k}) \mathcal{R}(\omega_m(\mathbf{k})/\omega_c)\ k\ dk. \tag{7}$$

The integrand in the above expression is a mix of both $\omega_m(\mathbf{k})$ and k. To facilitate the integration, therefore, let us express the integrand in terms of $\omega_m(\mathbf{k})$ alone. Recall from (1) that

$$\omega_m(\mathbf{k}) = c \left(\frac{m^2\pi^2}{r^2} + k^2 \right)^{1/2}.$$

[6] As can be inferred from the argument given here, in a general dimension D, $\sum_{\mathbf{k}} \to \left[\frac{l_1 l_2 \ldots l_D}{(2\pi)^D} \right] \int d^D k$.

Thus,

$$k = 0 \implies \omega_m(\mathbf{0}) = cm\pi/r, \;\; k \to \infty \implies \omega_m(\mathbf{k}) \to \infty \;\; \forall m.$$

We therefore have established the new limits of integration in terms of the angular frequency of the waves. Since

$$\omega_m^2(\mathbf{k}) = c^2 \left(\frac{m^2\pi^2}{r^2} + k^2 \right),$$

for a given m,

$$2k\; dk = \frac{1}{c^2}[2\omega_m(\mathbf{k})\; d\omega_m(\mathbf{k})] \implies k\; dk = \frac{1}{c^2}[\omega_m(\mathbf{k})\; d\omega_m(\mathbf{k})].$$

Therefore, in terms of $\omega_m(\mathbf{k})$, along with the form of the regularization function [see (4)], (7) can be written as

$$\mathcal{E}_{0_{\text{in}}} = \frac{A\hbar}{2\pi c^2} \sum_{m=[0]}^{\infty} \int_{\omega_m(\mathbf{0})}^{\infty} \omega_m^2(\mathbf{k}) \exp[-\omega_m(\mathbf{k})/\omega_c]\; d\omega_m(\mathbf{k}), \;\; \omega_m(\mathbf{0}) = cm\pi/r. \tag{8}$$

To evaluate the above integral, we can utilize the method of integration by parts.[7] Noting that

$$\int \exp[-\omega_m(\mathbf{k})/\omega_c]\; d\omega_m(\mathbf{k}) = -\omega_c \;\exp[-\omega_m(\mathbf{k})/\omega_c],$$

and using integration by parts twice to eliminate the $\omega_m^2(\mathbf{k})$ term, we arrive at

$$\boxed{\mathcal{E}_{0_{\text{in}}} = \frac{A\hbar}{2\pi c^2} \sum_{m=[0]}^{\infty} \left[\omega_c\omega_m^2(\mathbf{0}) + 2\omega_c^2\omega_m(\mathbf{0}) + 2\omega_c^3\right] \exp[-\omega_m(\mathbf{0})/\omega_c], \;\; \omega_m(\mathbf{0}) = cm\pi/r.} \tag{9}$$

This is the (regularized) energy in the vacuum in the space bounded by the plates.

2.5 The force on the plates

Let us now consider the force on the plates due to the (regularized) energy in the vacuum within the space bounded by the plates. This is obtained by taking the derivative of (9) with respect to the distance r between the plates. To compute this derivative, let us first write (9) in terms of r, which is

[7] An integration by parts takes on the form $\int_a^b u \frac{d}{dx} v\; dx = uv|_a^b - \int_a^b v \frac{d}{dx} u\; dx$, where u and v are functions of x.

$$\mathcal{E}_{0_{\rm in}} = \frac{A\hbar}{2\pi c^2} \sum_{m=[0]}^{\infty} \left[\omega_c (cm\pi)^2 r^{-2} + 2\omega_c^2 (cm\pi) r^{-1} + 2\omega_c^3\right] \exp[-(cm\pi/\omega_c) r^{-1}]. \tag{10}$$

Then, the force ($F_{\rm in}$) on the plates due to the vacuum in the space bounded by the plates is

$$F_{\rm in} = -\frac{\partial}{\partial r}\mathcal{E}_{0_{\rm in}} = -\frac{A\hbar}{2\pi c^2} \sum_{m=[0]}^{\infty} \frac{\partial}{\partial r} \left\{ \left[\omega_c (cm\pi)^2 r^{-2} + 2\omega_c^2 (cm\pi) r^{-1} + 2\omega_c^3\right] \exp[-(cm\pi/\omega_c) r^{-1}] \right\}.$$

The derivatives result in expressions having positive terms proportional to $1/r^2$ and $1/r^3$ which cancel with their respective negative terms leaving only a term proportional to $1/r^4$, giving

$$F_{\rm in} = -\frac{A\hbar}{2\pi c^2} \sum_{m=[0]}^{\infty} \frac{c^3 m^3 \pi^3}{r^4} \exp[-(cm\pi/\omega_c) r^{-1}].$$

Thus,

$$F_{\rm in} = -\frac{A\pi^2 \hbar c}{2r^4} \sum_{m=[0]}^{\infty} m^3 \exp\left(-\frac{c\pi}{\omega_c r} m\right). \tag{11}$$

We note here, that except for the regularization term $\exp[-c\pi m/(w_c r)]$, the force on the plates from the vacuum within is given by the summation of m^3 terms, each m being a discrete wave mode reflecting between the parallel plates. Recall that the symbol $m = [0]$ signifies the need to insert a $1/2$ factor in the sum when $m = 0$. However, as can be seen from (11), the $1/2$ factor is rendered mute due to the m^3 term since the summand vanishes when $m = 0$. Thus, the sum over m in (11) can now be effectively regarded as not requiring the $1/2$ factor to be included when $m = 0$. Therefore, (11) can be written as

$$\boxed{F_{\rm in} = -\frac{A\pi^2 \hbar c}{2r^4} \sum_{m=0}^{\infty} m^3 \exp\left(-\frac{c\pi}{\omega_c r} m\right). \quad \text{force on the plates from the vacuum within}} \tag{12}$$

In contrast to (11), the form of (12) is of importance since it implies that we can directly convert the discrete sum over m (ranging from 0 to ∞ without the need for the $1/2$ factor when $m = 0$) into a continuous integral over m with the integration limits ranging from 0 to ∞. This need will arise shortly as will be seen below. We have now come to the end of our considerations of the vacuum effect on the plates from the space within.

Considering the force ($F_{\rm out}$) on the plates from the rest of the vacuum not bounded by the plates, it seems reasonable to consider waves reflecting against the outer side of the left plate, say, with a hypothetical parallel plate to the left of it placed at infinity; a similar scenario can be envisioned for the plate on the right. Effectively, this amounts to considering the waves reflecting between the parallel plates when $r \to \infty$. Therefore, we posit that

$$F_{\text{out}} = \lim_{r\to\infty} F_{\text{in}}.$$

The wave modes reflecting between parallel plates placed an infinite distance apart can no longer be considered as discrete but continuous. Hence, to compute F_{out} we treat m as continuous, and as such, the sum over m in (12) becomes an integral over m. Thus

$$\boxed{F_{\text{out}} = -\frac{A\pi^2\hbar c}{2r^4}\int_0^\infty m^3 \exp\left(-\frac{c\pi}{\omega_c r}m\right) dm. \ \text{force on the plates from the vacuum outside}} \tag{13}$$

The net force on the plates is then

$$F_{\text{net}}|_{T=0} = \lim_{\omega_c\to\infty}(F_{\text{in}} - F_{\text{out}}),$$

where we have taken the frequency cut-off to tend to infinity to remove its dependence on the final observable, which is the net force.[8] Inserting the expressions for F_{in} and F_{out}, we have

$$\boxed{F_{\text{net}}|_{T=0} = -\frac{A\pi^2\hbar c}{2r^4}\lim_{\omega_c\to\infty}\left\{\sum_{m=0}^{\infty} m^3 \exp\left(-\frac{c\pi}{\omega_c r}m\right) - \int_0^\infty m^3 \exp\left(-\frac{c\pi}{\omega_c r}m\right) dm\right\}.} \tag{14}$$

The expression in the curly bracket is what we have alluded to in the introductory part of this section: it is the difference of a sum of a function and its integral. The sum and the integral, each taken individually, is infinite in the limit $\omega_c \to \infty$.[9] However, their difference is finite as we can demonstrate by using the Euler-Maclaurin summation.[10] For a function $f(m)$ with the lower and upper limits of our interest, the Euler-Maclaurin summation takes the form

[8]Taking the difference $F_{\text{in}} - F_{\text{out}}$ reflects the fact that it is the energy difference in the space bounded within and outside of the plates that gives rise to the net force so that $F_{\text{net}} = -[\frac{\partial}{\partial r}(\mathcal{E}_{0_{\text{in}}} - \mathcal{E}_{0_{\text{out}}})]$.

[9]$\lim_{\omega_c\to\infty}\sum_{m=0}^{\infty} m^3 \exp\left(-\frac{c\pi}{\omega_c r}m\right) = \sum_{m=0}^{\infty} m^3 \to \infty$; $\lim_{\omega_c\to\infty}\int_0^\infty m^3 \exp\left(-\frac{c\pi}{\omega_c r}m\right) dm = \lim_{\omega_c\to\infty}[6(\omega_c r/(c\pi))^4] \to \infty$.

[10]Readers who have read popular accounts of quantum field theory may recall the idea of renormalization being introduced as infinities "cancelling" each other resulting in finite answers. Here we have an actual instantiation of infinities "cancelling" in a relatively simpler setting. For details on the Euler-Maclaurin summation see *Divergent Series* (Oxford, 1949) by G. H. Hardy. For more mathematical results regarding the Casimir effect, the reader is referred to the paper by Jonathan P. Dowling, *The Mathematics of the Casimir Effect*, Mathematics Magazine, Vol. 62, No.5, December, 1989, 324-331.

$$\sum_{m=0}^{\infty} f(m) - \int_0^{\infty} f(m)\, dm = \frac{f(\infty)+f(0)}{2} + \frac{1}{6}\frac{f^{(1)}(\infty)-f^{(1)}(0)}{2!} - \frac{1}{30}\frac{f^{(3)}(\infty)-f^{(3)}(0)}{4!} + \cdots,$$

where $f^{(p)}$ refers to the p-th derivative of the function with respect to m. Given that the function of interest here is

$$f(m) = m^3 \exp\left(-\frac{c\pi}{\omega_c r} m\right),$$

it is evident that $f(0) = 0$ and $f^{(1)}(0) = 0$. Considering $f(\infty)$, we note that the exponential term in $f(m)$ tends to zero faster than its cubic term tends to infinity when $m \to \infty$. Thus, $f(\infty) = 0$. Similarly, $f^{(1)}(\infty) = 0$. Now,

$$f^{(3)}(m) = 6 \exp\left(-\frac{c\pi}{\omega_c r} m\right) + [\sim (m + m^2) \exp(-m) + (m^2 + m^3) \exp(-m)].$$

Therefore, $f^{(3)}(0) = 6$ and $f^{(3)}(\infty) = 0$. In fact, every $f^{(p)}(\infty) = 0,\ p \in \mathbb{N}$. The $f^{(p)}(0)$ terms for $p \geq 5$ will have terms of the order $(1/w_c)^2$ or higher; such terms will therefore vanish when $\omega_c \to \infty$. In essence, we therefore have

$$\lim_{\omega_c \to \infty} \left\{ \sum_{m=0}^{\infty} m^3 \exp\left(-\frac{c\pi}{\omega_c r} m\right) - \int_0^{\infty} m^3 \exp\left(-\frac{c\pi}{\omega_c r} m\right) dm \right\} = -\frac{1}{30} \cdot \frac{1}{4!}(0-6) = \frac{1}{5!} = \frac{1}{120}.$$

Inserting this number in (14) we finally obtain the net force on the plates due to the vacuum at absolute zero temperature as

$$\boxed{F_{\text{net}}|_{T=0} = -A\frac{\pi^2}{240}\frac{\hbar c}{r^4}. \text{ net attractive force on the plates due to the vacuum at } T = 0\ K} \tag{15}$$

The negative sign in the above expression indicates that the two plates are attracted to each other due to vacuum fluctuations. This is the Casimir effect, which, as we can see, varies as the inverse fourth-power of the distance between the plates at absolute zero temperature. The force between the plates is also directly proportional to the surface area of a plate. We can note here that taking the difference between the sum and the integral through the Euler-Maclaurin summation removes the dependency of the observable force on the cut-off frequency in the limit $\omega_c \to \infty$, thereby sparing us of setting some arbitrary finite cut-off. Thus, the regularization imposed to tame the raw infinity of the vacuum energy results, in the end, in a finite answer for the observable force with no trace of the regularization frequency cut-off. Continuing, the pressure exerted on a plate by the vacuum energy at absolute zero is then given by

$$P|_{T=0} = \frac{|F_{\text{net}}|_{T=0}}{A} = \frac{\pi^2}{240}\frac{\hbar c}{r^4}. \quad \text{pressure on a plate due to the vacuum at } T = 0\ K \tag{16}$$

Let us keep one plate fixed and attempt to move the other plate to infinity against the attractive force. (The plates are kept parallel to each other at all times.) The energy required to accomplish this task can be computed as follows: Let us assume that the plate on the left is fixed at the origin of the coordinate system (see Figure 2) and let us take the unit vector in the positive x-direction as $\mathbf{i}$ (so $\mathbf{i} \cdot \mathbf{i} = 1$). Then the applied force on the plate on the right, which is moved in the positive x-direction is $\mathbf{F} = |F_{\text{net}}|_{T=0}\mathbf{i}$, where, if the separation between the plates at any given instant is x, $|F_{\text{net}}|_{T=0} = A\pi^2\hbar c/(240x^4)$. Let us also assume that the initial separation between the plates is r. Thus, the work done (W) or the energy expended on the system in separating the plates an infinite distance apart at absolute zero temperature is,

$$W|_{T=0} = \int_r^\infty \mathbf{F} \cdot d\mathbf{x} = \int_r^\infty |F_{\text{net}}|_{T=0}\, dx(\mathbf{i} \cdot \mathbf{i}) = A\frac{\pi^2}{240}\hbar c \int_r^\infty \frac{1}{x^4}dx = A\frac{\pi^2}{720}\frac{\hbar c}{r^3}. \tag{17}$$

To extend the meaning of the result (17), we infer that the net force given in the expression (15) arises due to the difference in the vacuum energy inside and outside the space bounded by the plates. To see this, note that as the separation between the plates widen, the net force on them decreases (as $1/r^4$) so that at a separation of infinity, there is no attractive force between the plates due to vacuum energy. In other words, the initial difference in vacuum energy that existed when the plates were a distance r apart disappears when they are an infinite distance apart. This means, the work done in separating the plates an infinite distance apart must equal the initial energy difference (in magnitude). If the energy difference is denoted as $\Delta\mathcal{E}_0$, then

$$\Delta\mathcal{E}_0 = \mathcal{E}_{0_{\text{in}}} - \mathcal{E}_{0_{\text{out}}}, \quad \text{where} \quad \mathcal{E}_{0_{\text{in}}} < \mathcal{E}_{0_{\text{out}}}.$$

The inequality $\mathcal{E}_{0_{\text{in}}} < \mathcal{E}_{0_{\text{out}}}$ is inferred from the fact that the force between the plates is attractive, and hence, the vacuum energy outside the plates must be larger than the vacuum energy in the space bounded by the plates (to apply a net force inward). The work ($W > 0$) done in separating the plates to infinity is against the attractive force, and hence, is against the energy difference $\Delta\mathcal{E}_0$. Thus,

$$\Delta\mathcal{E}_0 = \mathcal{E}_{0_{\text{in}}} - \mathcal{E}_{0_{\text{out}}} = -W = -A\frac{\pi^2}{720}\frac{\hbar c}{r^3}. \tag{18}$$

The net force arising due to this energy difference at absolute zero temperature is then

$$F_{\text{net}}|_{T=0} = -\frac{\partial}{\partial r}\Delta\mathcal{E}_0 = A\frac{\pi^2}{720}\hbar c\left(\frac{\partial}{\partial r}\frac{1}{r^3}\right) = -A\frac{\pi^2}{240}\frac{\hbar c}{r^4}.$$

We have thus obtained the same result as in (15).

3 Casimir Effect at Very Low Temperature

In this section our goal is to obtain insights on the Casimir effect at temperatures just above the absolute zero. This is important since, according to the third law of thermodynamics, absolute zero temperature is unattainable in an experiment. In this case, in addition to the zero-point energy discussed in the previous section, we will have to account for the thermal energy of the space (which we will still call the vacuum) within which the plates are placed. Above absolute zero there will be real photons contributing to the thermal energy of the vacuum. If a photon is in the angular frequency mode $\omega_m(\mathbf{k})$, and if there are n number of such photons in the vacuum, then the Hamiltonian ($\mathcal{H}_{n,m}(\mathbf{k})$) of the vacuum is given by,

$$\mathcal{H}_{n,m}(\mathbf{k}) = \left(n + \frac{1}{2}\right)\hbar\omega_m(\mathbf{k}), \quad n,\ m \in \mathbb{N}.$$

In this expression, as before, the half-factor accounts for the zero-point energy. The thermal energy part is contained in $n\hbar\omega_m(\mathbf{k})$, where n is an integer in the range $[0, \infty)$. Thus, for any given angular frequency mode $\omega_m(\mathbf{k})$, there can be any number of photons. For an absolute temperature of T, the Boltzmann factor ($\mathcal{B}$) associated with $\mathcal{H}_{n,m}(\mathbf{k})$ is:

$$\mathcal{B} = \exp\left[-\beta\mathcal{H}_{n,m}(\mathbf{k})\right], \quad \beta = \frac{1}{k_B T}, \quad k_B = \text{Boltzmann constant.}$$

The probability of finding the vacuum in the state characterized by $\mathcal{H}_{n,m}(\mathbf{k})$ with n photons (in a given angular frequency mode $\omega_m(\mathbf{k})$) is proportional to the Boltzmann factor. If the proportionality constant is $1/Z$, then Z is a normalization factor. Since the vacuum must contain either no photons, or a single photon, or two photons, etc., in the angular frequency mode $\omega_m(\mathbf{k})$, the sum of all these probabilities over the number of photons must be unity. This, then, implies that

$$Z = \sum_{n=0}^{\infty} \exp\left[-\beta H_{n,m}(\mathbf{k})\right] = \sum_{n=0}^{\infty} \exp\left[-\beta\left(n + \frac{1}{2}\right)\hbar\omega_m(\mathbf{k})\right] = \exp\left[-\frac{1}{2}\beta\hbar\omega_m(\mathbf{k})\right] \cdot \sum_{n=0}^{\infty} \exp\left[-n\beta\hbar\omega_m(\mathbf{k})\right].$$

The normalization factor Z is known as the partition function, which is one of the most fundamental entities in statistical mechanics from which all of the thermodynamics of a system in equilibrium follow. Now, for any $n \neq 0$, $\exp[-n\beta\hbar\omega_m(\mathbf{k})] < 1$. Note that if a real photon exists, that is, if $n \neq 0$, then $\omega_m(\mathbf{k}) > 0$; that is, if a real photon exists, it must have some angular frequency, which, according to (1), implies that both m and k cannot simultaneously be zero (β and $\hbar$ are both positive, of course). Therefore, since $\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right] < 1$ for real photons,

$$\begin{aligned}\sum_{n=0}^{\infty}\exp\left[-n\beta\hbar\omega_m(\mathbf{k})\right] &= 1+\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right]+\exp\left[-2\beta\hbar\omega_m(\mathbf{k})\right]+\cdots,\\ &= 1+\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right]+\left(\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right]\right)^2+\cdots,\\ &= \frac{1}{1-\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right]}.\end{aligned}$$

The partition function now reads

$$Z=\exp\left[-\frac{1}{2}\beta\hbar\omega_m(\mathbf{k})\right]\cdot\left(1-\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right]\right)^{-1}.$$

Since the free energy ($\mathcal{E}_F$) of a system, by definition, is $\mathcal{E}_F=-\frac{1}{\beta}\ln Z$, the free energy of the vacuum (accounting only a single polarization state of a photon) is

$$\mathcal{E}_F=\frac{1}{2}\hbar\omega_m(\mathbf{k})+\frac{1}{\beta}\ln\left(1-\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right]\right).$$

Thus, considering both polarization states of a photon, the total free energy ($\mathcal{F}$) of the vacuum is given by twice that of the above expression:

$$\boxed{\mathcal{F}=\hbar\omega_m(\mathbf{k})+\frac{2}{\beta}\ln\left(1-\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right]\right),\quad \exp[-\beta\hbar\omega_m(\mathbf{k})]<1.} \tag{19}$$

In (19) we recognize the first term on the right as representing the zero-point energy of the vacuum at absolute zero temperature, which we analyzed in the previous section. The second term on the right relates to the thermal energy of the vacuum at a given non-zero absolute temperature. Our goal in this section is to analyze the consequences that this thermal energy term gives rise to.

With the goal set out in the previous paragraph in mind, we recognize (as in the previous section), that we have to sum the second term on the right hand side of (19) over the modes m and the wave vectors $\mathbf{k}$ to obtain the thermal energy ($\mathcal{E}_{T_{\text{in}}}$) in the vacuum bounded by the plates. Thus,

$$\mathcal{E}_{T_{\text{in}}}=\frac{2}{\beta}\sum_{m,\mathbf{k}}\ln\left(1-\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right]\right),\quad \exp[-\beta\hbar\omega_m(\mathbf{k})]<1. \tag{20}$$

To keep the notation manageable, let us call

$$\ln\left(1-\exp\left[-\beta\hbar\omega_m(\mathbf{k})\right]\right)=\Theta(\beta\hbar\omega_m(\mathbf{k})),$$

whereby (20) takes the form,

$$\mathcal{E}_{T_{\text{in}}} = \frac{2}{\beta} \sum_{m,\mathbf{k}} \Theta(\beta\hbar\omega_m(\mathbf{k})) = \frac{2}{\beta} \sum_{m} \sum_{\mathbf{k}} \Theta(\beta\hbar\omega_m(\mathbf{k})).$$

Then, by the same argument that lead to (6), we have

$$\mathcal{E}_{T_{\text{in}}} = \frac{2}{\beta} \frac{A}{(2\pi)^2} \sum_{m=1}^{\infty} \int d^2k \; \Theta(\beta\hbar\omega_m(\mathbf{k})) = \frac{2}{\beta} \frac{A}{(2\pi)^2} \sum_{m=1}^{\infty} \int_0^{2\pi} d\phi \int_0^{\infty} \Theta(\beta\hbar\omega_m(\mathbf{k})) \; k \; dk.$$

Note that in the above expression we have taken the minimum value of m to be 1 so that both k and m will not vanish simultaneously. More justification for this follows below. Therefore,

$$\mathcal{E}_{T_{\text{in}}} = \frac{A}{\pi\beta} \sum_{m=1}^{\infty} \int_0^{\infty} \Theta(\beta\hbar\omega_m(\mathbf{k})) \; k \; dk,$$

where A is the inner area of a plate. Recalling from (1) that $k\;dk = \omega_m(\mathbf{k})\; d\omega_m(\mathbf{k})/c^2$ and that $\omega_m(\mathbf{0})) = cm\pi/r$, the above expression can be recast as

$$\mathcal{E}_{T_{\text{in}}} = \frac{A}{\pi\beta c^2} \sum_{m=1}^{\infty} \int_{\omega_m(\mathbf{0}))}^{\infty} \Theta(\beta\hbar\omega_m(\mathbf{k})) \; \omega_m(\mathbf{k}) \; d\omega_m(\mathbf{k}).$$

Note that given the form of (20) where $\exp[-\beta\hbar\omega_m(\mathbf{k})] < 1$, we expect the series to converge, and therefore, unlike in the absolute zero temperature case, there is no need to introduce a regularization.

Let us now make another variable change as follows: let $\beta\hbar\omega_m(\mathbf{k}) = p_m(\mathbf{k})$. Since $\beta\hbar\omega_m(\mathbf{k})$ is dimensionless, $p_m(\mathbf{k})$ is also a dimensionless variable. Then, $d\omega_m(\mathbf{k}) = dp_m(\mathbf{k})/(\beta\hbar)$. Also, when $\mathbf{k} = \mathbf{0}$, $\beta\hbar\omega_m(\mathbf{0}) = p_m(\mathbf{0})$. But $\omega_m(\mathbf{0}) = cm\pi/r$. Therefore, $p_m(\mathbf{0}) = (\pi\beta\hbar c/r)m = p_m$. In order to satisfy the condition $\exp[-\beta\hbar\omega_m(\mathbf{k})] = \exp[-p_m(\mathbf{k})] < 1 \; \forall \mathbf{k}$, it must be the case that $m \neq 0$, for otherwise, p_m would be zero when $m = 0$ leading to $\exp[-p_m(\mathbf{0})]|_{m=0} = 1$. The above expression for $\mathcal{E}_{T_{\text{in}}}$ then takes the form

$$\mathcal{E}_{T_{\text{in}}} = \frac{A}{\pi\beta} \frac{1}{(\beta\hbar c)^2} \sum_{m=1}^{\infty} \int_{p_m}^{\infty} \Theta(p_m(\mathbf{k})) \; p_m(\mathbf{k}) \; dp_m(\mathbf{k}), \tag{21}$$

where

$$\Theta(p_m(\mathbf{k})) = \ln\left(1 - \exp\left[-p_m(\mathbf{k})\right]\right), \quad p_m = \frac{\pi\beta\hbar c}{r} m = p_m(\mathbf{0}), \quad m \in \mathbb{N}_+. \tag{22}$$

Now, let

$$\int_{p_m}^{\infty} \Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\ dp_m(\mathbf{k}) = \chi(p_m). \tag{23}$$

Substituting this in (21), the thermal energy in the vacuum bounded by the plates is given by the expression

$$\boxed{\mathcal{E}_{T_{\text{in}}} = \frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\sum_{m=1}^{\infty}\chi(p_m), \quad \chi(p_m) = \int_{p_m}^{\infty}\Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\ dp_m(\mathbf{k}).} \tag{24}$$

The above expression cannot be directly evaluated due to p_m (which depends on m) appearing as the lower limit of the integral followed by the sum over m. Therefore, keeping this form as is, let us compute the force resulting from the thermal energy within the vacuum bounded by the plates, which is

$$F_{T_{\text{in}}} = -\frac{\partial}{\partial r}\mathcal{E}_{T_{\text{in}}} = -\frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\sum_{m=1}^{\infty}\frac{\partial}{\partial r}\chi(p_m) = -\frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\sum_{m=1}^{\infty}\frac{\partial}{\partial r}\left[\int_{p_m}^{\infty}\Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\ dp_m(\mathbf{k})\right].$$

We now have a differentiation of an integral which can be handled by the Leibniz's rule, which reads

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,y)\ dy = f(x,b(x))\frac{\partial}{\partial x}b(x) - f(x,a(x))\frac{\partial}{\partial x}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,y)\ dy.$$

Applied to the specific case at hand, where r plays the role of x, $p_m(\mathbf{k})$ plays the role of y and $\Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})$ plays the role of $f(x,y)$ with $a(x) = p_m$ and $b(x) = \infty$, we have

$$\begin{aligned}\frac{\partial}{\partial r}\int_{p_m}^{\infty}\Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\ dp_m(\mathbf{k}) = \Theta(\infty)\cdot(\infty)\cdot\frac{\partial}{\partial r}(\infty) \\ -\Theta(p_m)\cdot p_m\cdot\frac{\partial}{\partial r}p_m \\ +\int_{p_m}^{\infty}\frac{\partial}{\partial r}\left[\Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\right]dp_m(\mathbf{k}).\end{aligned} \tag{25}$$

From the definition of $\Theta(p_m(\mathbf{k}))$ in (22), it follows that $\Theta(\infty) = \ln(1) = 0$; therefore, the first term on the right hand side of (25) vanishes. To evaluate the third term on the right hand side of (25) let us make the assumption that the temperature of the vacuum, although non-zero, is quite small. In that case, since $p_m = \pi\beta\hbar cm/r$, and therefore $p_m \propto 1/T$, for very small T, p_m is very large. If we take this $p_m \to \infty$, then the third term on the right hand side of (25) vanishes since whatever the integrand, the limits of integration become the same. Therefore, for very small temperatures ($T = 0_+$), it is effectively the case that,

$$\left[\frac{\partial}{\partial r}\int_{p_m}^{\infty} \Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\ dp_m(\mathbf{k})\right]_{T=0_+} = -\Theta(p_m)\cdot p_m \cdot \frac{\partial}{\partial r}p_m.$$

Since $p_m = \pi\beta\hbar cm/r$,

$$\frac{\partial}{\partial r}p_m = -\frac{\pi\beta\hbar cm}{r^2} = -\frac{p_m}{r}.$$

Therefore,

$$\left[\frac{\partial}{\partial r}\int_{p_m}^{\infty} \Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\ dp_m(\mathbf{k})\right]_{T=0_+} = \frac{1}{r}\cdot p_m^2 \cdot \Theta(p_m).$$

Substituting this in the expression for the force above, we obtain, for very low temperatures,

$$F_{T_{\mathrm{in}[0+]}} = -\frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{r}\sum_{m=1}^{\infty} p_m^2 \cdot \Theta(p_m). \tag{26}$$

Now, since

$$\Theta(p_m(\mathbf{k})) = \ln\left(1 - \exp\left[-p_m(\mathbf{k})\right]\right) \implies \Theta(p_m) = \ln\left(1-\exp[-p_m]\right),\ \ \exp(-p_m) < 1,$$

for very small temperatures (that is, for very large p_m, and therefore, for very small $\exp(-p_m)$) it follows that[11]

$$\Theta(p_m)\Big|_{T=0+} \approx -\exp(-p_m).$$

Therefore,

$$\boxed{F_{T_{\mathrm{in}[0+]}} = \frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{r}\sum_{m=1}^{\infty} p_m^2 \cdot \exp(-p_m)\ \text{ at very low } T.} \tag{27}$$

Now, let

[11] For any $|\alpha| < 1$, $\ln(1+\alpha) = \frac{\alpha}{1} - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \cdots \implies \ln(1-\alpha) = -\left(\frac{\alpha}{1} + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \cdots\right)$. For very small α all but the first term in the series for $\ln(1-\alpha)$ can be ignored. Thus, $\ln(1-\alpha) \approx -\alpha$.

$$\gamma = \frac{\pi\beta\hbar c}{r}.$$

Then,

$$p_m = \frac{\pi\beta\hbar c}{r} m = \gamma m, \quad m \in \mathbb{N}_+.$$

Equation (27) can then be written as

$$F_{T_{\text{in}[0+]}} = \frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{r}\sum_{m=1}^{\infty}\gamma^2 m^2 \exp(-\gamma m) = \frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{r}\gamma^2\left[\exp(-\gamma) + 4\exp(-2\gamma) + \cdots\right].$$

Since γ is large at low T, we can ignore $\exp(-2\gamma) = [\exp(-\gamma)]^2$ and higher terms. Thus,

$$\boxed{F_{T_{\text{in}[0+]}} \approx \frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{r}\gamma^2 \exp(-\gamma), \quad \gamma = \frac{\pi\beta\hbar c}{r}, \quad \beta = \frac{1}{k_B T}; \quad \text{at very low } T.} \tag{28}$$

The force on the plates due to thermal energy of the vacuum bounded by the plates at very low temperatures is therefore given by (28). Since the expression is a positive quantity, this implies that the thermal energy of the vacuum bounded by the plates exerts a repulsive force on the plates. Furthermore, to the lowest order in $\exp(-\gamma)$ (which is 1) this repulsive force is proportional to T/r^3. Thus, as expected, this force component due to thermal energy vanishes at $T = 0$ K.

We now consider the thermal energy of the vacuum outside the plates. This is akin to considering $\mathcal{E}_{T_{\text{in}}}$ in the limit $r \to \infty$; that is, when the separation between the parallel plates is infinite. In this limit the energy difference between photons in adjacent modes (e.g., m vs. $m+1$) becomes smaller than $k_B T$. We can therefore consider m as continuous starting at $m = 0$ to reflect the vastness of the distance between a plate and the rest of the space outside of it. An analogy is a string of finite length with some transverse movement; if this string is to be stretched out to infinity, then we can imagine the vanishing of its transverse movement. This would correspond to the $m = 0$ mode of vibration of the (thermal) electromagnetic wave, and by extension, to its $p_m = 0$ state. However, in order to satisfy the condition $\exp(-p_m) < 1$, small but non-zero limits for m and p_m must be imposed in the integrals. Let us denote the corresponding small but non-zero limits as $\mathfrak{m}$ and $\mathfrak{p}$, respectively. Once the integrals are evaluated, we can let $\mathfrak{m}, \mathfrak{p} \to 0$ to obtain the final results. Bearing these thoughts, we can now spring-board from (24) and write

$$\mathcal{E}_{T_{\text{out}}} = \frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\lim_{\mathfrak{m}\to 0}\int_{\mathfrak{m}}^{\infty}\chi(p_m)\,dm. \tag{29}$$

Since $dm = [r/(\pi\beta\hbar c)]\ dp_m$, we can write the above as

$$\mathcal{E}_{T_{\text{out}}} = \frac{Ar}{\pi^2\beta}\frac{1}{(\beta\hbar c)^3}\lim_{\mathfrak{p}\to 0}\int_{\mathfrak{p}}^{\infty}\chi(p_m)\ dp_m. \tag{30}$$

Using integration by parts, this integral can be cast as

$$\int_{\mathfrak{p}}^{\infty}\chi(p_m)\ dp_m = \int_{\mathfrak{p}}^{\infty}\chi(p_m)\frac{dp_m}{dp_m}\ dp_m = \chi(p_m)\ p_m\Big|_{\mathfrak{p}}^{\infty} - \int_{\mathfrak{p}}^{\infty} p_m\frac{d\chi(p_m)}{dp_m}\ dp_m.$$

Since

$$\chi(p_m) = \int_{p_m}^{\infty}\Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\ dp_m(\mathbf{k}) \implies \chi(\infty) = \int_{\infty}^{\infty}\Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\ dp_m(\mathbf{k}) = 0$$

due to having same limits on the integral,

$$\chi(p_m)\ p_m\Big|_{\mathfrak{p}}^{\infty} = \chi(\infty)\cdot(\infty) - \chi(\mathfrak{p})\cdot(\mathfrak{p}) = 0 - \chi(\mathfrak{p})\cdot(\mathfrak{p}) = -\chi(\mathfrak{p})\cdot(\mathfrak{p}).$$

We now let $\mathfrak{p}\to 0$. Thus,

$$\lim_{\mathfrak{p}\to 0}\chi(p_m)\ p_m\Big|_{\mathfrak{p}}^{\infty} = \lim_{\mathfrak{p}\to 0}[-\chi(\mathfrak{p})\cdot(\mathfrak{p})] = 0.$$

Therefore,

$$\lim_{\mathfrak{p}\to 0}\int_{\mathfrak{p}}^{\infty}\chi(p_m)\ dp_m = -\lim_{\mathfrak{p}\to 0}\int_{\mathfrak{p}}^{\infty} p_m\frac{d\chi(p_m)}{dp_m}\ dp_m.$$

Since

$$\chi(p_m) = \int_{p_m}^{\infty}\Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k})\ dp_m(\mathbf{k}) \implies \frac{d\chi(p_m)}{dp_m(\mathbf{k})} = \Theta(p_m(\mathbf{k}))\ p_m(\mathbf{k}),$$

$$\frac{d\chi(p_m)}{dp_m} = \lim_{\mathbf{k}\to\mathbf{0}}\frac{d\chi(p_m)}{dp_m(\mathbf{k})} = \Theta(p_m(\mathbf{0}))\ p_m(\mathbf{0}) = \Theta(p_m)\ p_m.$$

It therefore follows that

$$\lim_{\mathfrak{p}\to 0}\int_{\mathfrak{p}}^{\infty}\chi(p_m)\ dp_m = -\lim_{\mathfrak{p}\to 0}\int_{\mathfrak{p}}^{\infty} p_m^2\ \Theta(p_m)\ dp_m.$$

Equation (30) now takes the form

$$\mathcal{E}_{T_{\text{out}}} = -\frac{Ar}{\pi^2\beta}\frac{1}{(\beta\hbar c)^3}\lim_{\mathfrak{p}\to 0}\int_{\mathfrak{p}}^{\infty} p_m^2\, \Theta(p_m)\, dp_m. \tag{31}$$

To evaluate this integral, let us first recall that

$$\Theta(p_m(\mathbf{k})) = \ln\left(1-\exp\left[-p_m(\mathbf{k})\right]\right) \implies \Theta(p_m) = \ln\left(1-\exp[-p_m]\right), \;\; \exp(-p_m) < 1.$$

Therefore, since for any $|\alpha| < 1$

$$\ln(1+\alpha) = \frac{\alpha}{1} - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \cdots \implies \ln(1-\alpha) = -\left(\frac{\alpha}{1} + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \cdots\right) = -\sum_{s=1}^{\infty}\frac{\alpha^s}{s}.$$

Taking $\alpha = \exp(-p_m)$, the above implies that

$$\Theta(p_m) = \ln\left(1-\exp[-p_m]\right) = -\sum_{s=1}^{\infty}\frac{\exp(-sp_m)}{s}.$$

Equation (31) can therefore be written as

$$\mathcal{E}_{T_{\text{out}}} = \frac{Ar}{\pi^2\beta}\frac{1}{(\beta\hbar c)^3}\lim_{\mathfrak{p}\to 0}\int_{\mathfrak{p}}^{\infty} p_m^2 \left[\sum_{s=1}^{\infty}\frac{\exp(-sp_m)}{s}\right] dp_m. \tag{32}$$

The sum is absolutely convergent, converging to $-\ln\left(1-\exp[-p_m]\right)$ since $\exp(-p_m) < 1$. Thus, we can interchange the order of the sum and the integral, giving

$$\mathcal{E}_{T_{\text{out}}} = \frac{Ar}{\pi^2\beta}\frac{1}{(\beta\hbar c)^3}\sum_{s=1}^{\infty}\frac{1}{s}\left[\lim_{\mathfrak{p}\to 0}\int_{\mathfrak{p}}^{\infty} p_m^2\, \exp(-sp_m)\, dp_m\right]. \tag{33}$$

Noting that

$$\int \exp(-sp_m)\, dp_m = -\frac{1}{s}\exp(-sp_m),$$

the integral in (33) can be performed by applying integration by parts twice to eliminate the p_m^2 term. This results in

$$\lim_{\mathfrak{p}\to 0}\int_{\mathfrak{p}}^{\infty} p_m^2\, \exp(-sp_m)\, dp_m = \lim_{\mathfrak{p}\to 0}\, \exp(-s\mathfrak{p})\left(\frac{\mathfrak{p}^2}{s} + \frac{2\mathfrak{p}}{s^2} + \frac{2}{s^3}\right) = \frac{2}{s^3}.$$

Therefore,

$$\mathcal{E}_{T_{\text{out}}} = \frac{2Ar}{\pi^2\beta}\frac{1}{(\beta\hbar c)^3}\sum_{s=1}^{\infty}\frac{1}{s^4}.$$

We now note that

$$\sum_{s=1}^{\infty}\frac{1}{s^4} = \zeta(4) = \frac{\pi^4}{90},$$

where $\zeta(4)$ is the Riemann zeta function.[12] Thus, at absolute temperature T, the thermal energy of the vacuum outside the parallel plates, which are a distance r apart and has area A, is

$$\boxed{\mathcal{E}_{T_{\text{out}}} = \frac{2Ar}{\pi^2\beta}\frac{1}{(\beta\hbar c)^3}\zeta(4) = \frac{Ar}{\beta^4}\frac{1}{(\hbar c)^3}\frac{\pi^2}{45}, \quad \beta = \frac{1}{k_B T}, \quad T \neq 0.} \tag{34}$$

Note that the thermal energy of the vacuum outside the plates is inversely proportional to β^4, and therefore, is proportional T^4. This, of course, is the Stefan-Boltzmann law. Using this result, let us now obtain the corresponding force on the plates, which is given by

$$F_{T_{\text{out}}} = -\frac{\partial}{\partial r}\mathcal{E}_{T_{\text{out}}} = -\frac{2A}{\pi^2\beta}\frac{1}{(\beta\hbar c)^3}\zeta(4) = -\frac{A}{\beta^4}\frac{1}{(\hbar c)^3}\frac{\pi^2}{45}.$$

We can cast this expression in the form

$$\boxed{F_{T_{\text{out}}} = -\frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{r}\left(\frac{2\zeta(4)}{\gamma}\right), \quad \gamma = \frac{\pi\beta\hbar c}{r}, \quad \beta = \frac{1}{k_B T}, \quad T \neq 0.} \tag{35}$$

Thus, the force on the parallel plates due to thermal energy of the vacuum outside is given by (35). The minus sign indicates that the plates attract each other due to this force. Note that (35) is actually independent of the separation r between the plates since the product $r\gamma$ in the denominator does not contain r. The attractive force between the plates is proportional to T^4 as mentioned earlier. Note also, that unlike the force on the plates due to the vacuum inside the plates given by (28), the above expression is valid for all temperatures at or above absolute zero. If we lower the temperature down to absolute zero, then, as expected, both the thermal energy and the force due to that energy given by the expressions (34) and (35) vanish at $T = 0$ K. As

[12] See relevant texts or websites for various methods of evaluating the Riemann zeta function.

such the two expressions apply to the case $T = 0$ K as well.

The net force on the plates at very low temperatures can now be obtained using (28) and (35). Thus,

$$F_{T_{\text{net}[0+]}} = F_{T_{\text{in}[0+]}} + F_{T_{\text{out}}},$$

which gives[13]

$$F_{T_{\text{net}[0+]}} = -\frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{r}\left(\frac{2\zeta(4)}{\gamma} - \gamma^2\exp(-\gamma)\right), \quad \gamma = \frac{\pi\beta\hbar c}{r}, \quad \beta = \frac{1}{k_B T}; \quad \text{at very low } T. \tag{36}$$

We can infer that $2\zeta(4)/\gamma - \gamma^2\exp(-\gamma) > 0$. To see this, note that both these terms are monotonically decreasing functions of γ; the term $\gamma^2\exp(-\gamma)$ decreases faster than the term $2\zeta(4)/\gamma$ with increasing γ. Thus, their difference, $2\zeta(4)/\gamma - \gamma^2\exp(-\gamma)$, always remains positive. This leads to the conclusion that at very low temperatures, due to the overall negative sign in (36), the net force on the plates due to thermal energy in the vacuum is attractive. The inward pressure applied on the plates from the vacuum outside overwhelms the outward pressure applied on the plates from the vacuum inside due to thermal energy at very low temperatures. The thermal pressure on a plate is then,

$$P_{T_{[0+]}} = \frac{|F_{T_{\text{net}[0+]}}|}{A} = \frac{1}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{r}\left(\frac{2\zeta(4)}{\gamma} - \gamma^2\exp(-\gamma)\right), \quad \gamma = \frac{\pi\beta\hbar c}{r}, \quad \beta = \frac{1}{k_B T}; \quad \text{at very low } T. \tag{37}$$

Let us re-write (36) in the more elaborate form

$$F_{T_{\text{net}[0+]}} = -\underbrace{\frac{A\pi^2}{45(\hbar c)^3\beta^4}}_{\propto T^4} + \underbrace{\frac{A\pi\exp(-\pi\beta\hbar c/r)}{\beta r^3}}_{\propto T/r^3 \text{ to lowest order}}, \quad \beta = \frac{1}{k_B T}; \quad \text{at very low } T. \tag{38}$$

To obtain the total net force at very low temperatures, we need to add this to the net force resulting from the zero-point energy. (Recall that the free energy in (19) consists of two terms: one due to zero-point energy, and the other due to thermal energy.) Thus,

[13] Here we simply add $F_{T_{\text{in}[0+]}}$ and $F_{T_{\text{out}}}$ to obtain the net force since the positive sign of the former indicates a repulsive force on the plates and the negative sign of the latter indicates an attractive force. Thus, the vectorial directions of the forces are already accounted for in the signs.

$$F_{\text{net}}|_{T=0+} = F_{\text{net}}|_{T=0} + F_{T_{\text{net}[0+]}}.$$

Using the expressions (15) and (38), the total net force on the plates at very low temperatures is then given by

$$\boxed{F_{\text{net}}|_{T=0+} = -\frac{A\pi^2\hbar c}{240\, r^4} - \frac{A\pi^2}{45(\hbar c)^3\beta^4} + \frac{A\pi \exp(-\pi\beta\hbar c/r)}{\beta r^3}, \quad \beta = \frac{1}{k_B T}; \quad \text{at very low } T.} \tag{39}$$

At $T = 0$ the last two terms in the above expression vanish, leaving only the force due to zero-point energy difference of the vacuum (in the space between and outside the plates) intact.

4 The Quantum Vacuum

As we have seen, the Casimir effect is a fascinating phenomenon that has its origins in the quantum nature of the vacuum and how it "reacts" to the boundary conditions (e.g., parallel plates) placed in it. (The extent to which such a causal attribution can be made depends on the extent to which our models of the universe via physics and mathematics align with experimental evidence.) The vacuum fluctuations play a fundamental role in quantum field theory, which provide explanations for further phenomena such as the spontaneous emission of radiation by atoms and the Lamb shift (a fine splitting of energy levels in the hydrogen atom). The vacuum fluctuations near the boundary – that is, the event horizon – of a black hole give rise to Hawking radiation; here, the event horizon of the black hole may be considered as playing the role of the parallel plates in the Casimir effect, separating the vacuum to an inside and an outside (of the black hole). In the computations carried out earlier, divergent series that were considered as lacking in meaning by mathematicians of the past become meaningful in the context of vacuum fluctuations, yielding finite results such as the force on the plates at absolute zero temperature.

As an additional observation, consider virtual photons appearing from and disappearing into the void. If we associate a virtual photon with energy $+\hbar\omega$ appearing from the vacuum and $-\hbar\omega$ disappearing into the vacuum, then, at any instant, given that there is no restriction to the number of such photons, the total energy ($\mathcal{E}$) of the vacuum is

$$\mathcal{E} = \hbar\omega - \hbar\omega + \hbar\omega - \hbar\omega + \cdots = \hbar\omega(1 - 1 + 1 - 1 + \cdots).$$

Following G. H. Hardy (see the footnote 4 on page 4), how can the sum $1 - 1 + 1 - 1 + \cdots$ be defined? If we consider pairwise sums $(1-1)+(1-1)+\cdots$, we may argue that the sum is 0. However, if we write $1-1+1-1+\cdots = 1-(1-1+1-1+\cdots) = 1-[(1-1)+(1-1)+\cdots] = 1-[0]$, then we may argue that the sum is 1. Rather than being paradoxical, the series sum can be given a rigorous value, which is the average of 0 and 1. To see this, let the sum $1-1+1-1+\cdots = S$. Then,[14]

$$S = 1-1+1-1+\cdots = 1-(1-1+1-1+\cdots) = 1-(S) \implies 2S = 1 \implies S = 1/2.$$

Substituting this result in the above expression for energy, we obtain

$$\mathcal{E} = \frac{1}{2}\hbar\omega.$$

We recognize this result as the zero-point energy of the vacuum (for the given angular frequency ω of virtual photons). The 1/2 factor can be directly associated with the appearance from and the disappearance into the void of an infinity of virtual photons. This is an intriguing result that connects "divergent" series and virtual particles, bypassing the traditional approach of obtaining the same result using the elaborate methodologies of quantized harmonic oscillators or quantum field theory.

♣

[14]See page 6 of Hardy's book, *Divergent Series* (Oxford, 1949), for the axioms from which the sum follows.