The Condensed World of Bose and Einstein

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1 Purpose

In learning statistical mechanics it is more than likely that a sense of obviousness is conveyed about the ideas that the learners are then expected to readily absorb. However, an inspection of the historical evolution of the concepts shows that the pioneers themselves struggled to breath clarity into these notions over many years. An example is the famous phase space factor of Planck's law of blackbody radiation, $8\pi\nu^2/c^3$, where ν is the frequency of the radiation and c is the speed of light in a vacuum. This essay is an invitation for a closer look at some of these concepts anchoring around what is today known as the Bose-Einstein statistics and Bose-Einstein condensation.

In this essay, I first share a few historical notes regarding the origins of Bose-Einstein statistics and condensation. The phase space factor in Planck's law, $8\pi\nu^2/c^3$, has not been given a satisfactory quantum-theoretical derivation until the work of Bose. It was a revelation that Bose, in introducing his way of counting arrangements of photons in energy states, took it as an *assumption* that the phase space volume of a quantum state is h^3, which Einstein also assumes in his derivations following up on Bose's work. It was also a revelation that a proof of Bose-Einstein condensation (BEC) in a realistic setting was not achieved for more than seventy years after its prediction by Einstein in the context of ideal (that is, non-interactive) bose gases.[1] Interestingly, the somewhat brittle theoretical grounds in concluding the existence of BEC in an ideal bose gas can be gleaned when the sum over discrete energy states is taken to a continuous energy integral that runs from ground state energy to infinity, which then has to be interpreted as bosons only existing in excited states; these bosons in excited states then start to reach the ground state only after the system temperature falls below the critical temperature. While, in this essay, I also walk this somewhat shaky grounds in arriving at the BEC critical temperature, I have attempted to frame the narrative regarding the BEC transition in a more logical manner than is typically found in the literature.

My goal, therefore, is to present the physics of Bose-Einstein statistics and the basics of Bose-Einstein condensation in as much a self-contained way as is possible. As such, I place its theoretical development in its historical context for a better appreciation of its origins while attempting a pathway to an understanding, that I hope, is more illuminating than is commonly found. The interested reader is encouraged to pursue the references scattered within the essay for further enlightenment.

[1] See *Proof of Bose-Einstein Condensation for Dilute Trapped Gases* by E. Lieb and R. Seiringer (Phys. Rev. Lett. **88**, 170409, 2002). Their abstract reads: "The ground state of bosonic atoms in a trap has been shown experimentally to display Bose-Einstein condensation (BEC). We prove this fact theoretically for bosons with two-body repulsive interaction potentials in the dilute limit, starting from the basic Schrödinger equation; the condensation is 100% into the state that minimizes the Gross-Pitaevskii energy functional. *This is the first rigorous proof of BEC in a physically realistic, continuum model."* (My italics.) A bose gas is a gas consisting of bosons, which are particles with integer spin. The word *Boson* was first coined by P. Dirac in 1947.

2 A Beautiful Step Forward

The early beginnings in quantum physics lie in Max Planck's efforts to develop a theoretical basis for the spectrum of blackbody radiation. To match with experimental findings, Planck had to assume that the walls of a blackbody exchange energy in discrete packets with the electromagnetic radiation that it is in equilibrium with. It was an act of desperation for Planck who was a reluctant revolutionary of quantum theory.[2]

Planck's expression that describes the energy density of the blackbody radiation per unit frequency interval, $\rho(\nu, T)$, at frequency ν and absolute temperature T is

$$\rho(\nu, T) = \frac{8\pi\nu^2}{c^3} \cdot \frac{h\nu}{\exp(\frac{h\nu}{k_B T}) - 1},$$

where c is the speed of light in a vacuum, h is the Planck's constant, and k_B is the Boltzmann's constant. In deriving this formula, Planck assumed that the radiation field itself is not quantized and resorted to classical electrodynamics in obtaining the factor $8\pi\nu^2/c^3$. Ever since the above formula appeared, there were attempts to remove the ad hoc aspects of its derivation by Planck, Albert Einstein, Paul Ehrenfest, Wolfgang Pauli, and Peter Debye. However, in all these attempts, the first factor, $8\pi\nu^2/c^3$, emerged via classical electrodynamics by counting the number of modes of vibrations in the radiation field. Further, the second factor, $h\nu/\{\exp[h\nu/(k_B T)] - 1\}$, did not have a clear logical basis and its derivation was based on various elementary processes.

Satyendra Nath Bose began his work on Planck's formula after Meghnad Saha drew Bose's attention to two papers – one by Pauli, and the other by Einstein and Ehrenfest – both published in 1923. Both these papers attempted to generalize Einstein's 1917 derivation of Planck's law. Saha also drew Bose's attention to Arthur Compton's 1923 explanation of the scattering of X-rays, which the latter based on radiation quanta or photons.[3]

Recognizing the lack of logical harmony in the existing derivations, Bose was able to obtain half of the first factor – that is a factor of $4\pi\nu^2/c^3$ – by considering a minimum phase space volume, which he assumed to be h^3. In bringing $4\pi\nu^2/c^3$ to fully align with $8\pi\nu^2/c^3$, the historical record becomes unclear, at least in my view (I return to this aspect below). The second factor was derived by considering the number of ways in which to distribute the radiation quanta or photons over energy states.[4]

[2] See *Max Planck: the reluctant revolutionary* by H. Kragh, Physics World, December 2000, pp. 31-35.

[3] The word *Photon* for radiation quanta was introduced by G. N. Lewis in 1926.

[4] In deriving his radiation law, Planck considered distributing "energy elements" (quanta) over the resonators that he assumed made up a blackbody. In contrast, in deriving Planck's radiation law, Bose, in a sense, inverted Planck's argument and considered distributing quanta over energy states or phase space cells.

Having failed to get his derivation of Planck's radiation formula published in the *Philosophical Magazine*, Bose then wrote Einstein on 4 June 1924.[5]

Letter sent to Einstein by Bose on 4 June, 1924
(Einstein Archive, Doc. 6-127)

Respected Sir,
I have ventured to send you the accompanying article for your perusal and opinion. I am anxious to know what you think of it. You will see that I have tried to deduce the coefficient $8\pi\nu^2/c^3$ in Planck's Law independent of the classical electrodynamics, only assuming [that finite (crossed out in the letter)] that the ultimate elementary regions in the Phase-space has the content h^3. I do not know sufficient German to translate the paper. If you think the paper worth publication, I shall be grateful if you arrange for its publication in Zeitschrift für Physik. Though a complete stranger to you, I do not feel any hesitation in making such a request. Because we are all your pupils through your writings. I do not know whether you still remember that somebody from Calcutta asked your permission to translate your papers on Relativity in English. You acceded to the request. The book has since been published. I was the one who translated your paper on Generalised Relativity.

Yours faithfully
SN Bose

On 2 July 1924, Einstein responded to Bose:

Einstein's response to Bose on 2 July, 1924
(original postcard in German in Dhaka University Archives)

Dear Colleague!
I have translated your paper and given it to the Zeitschrift für Physik for publication. It signifies an important step forward and pleases me very much. However, I do not find your objection to my paper correct. For Wien's displacement law does not presuppose the wave theory and Bohr's correspondence principle is not used. But this is unimportant. You have derived the first factor quantum-theoretically even if not quite rigorously on account of the polarization factor 2. It is a beautiful step forward.

With friendly greetings
your A. Einstein

[5]See the article *Einstein and Bose* by J. Stachel (pp. 422–441) in *Satyendra Nath Bose, His Life and Times: Selected Works (with Commentary)* by K. C. Wali (Ed.), World Scientific, 2009. Many of the original papers by Bose along with other historical perspectives relevant to Bose's work are printed in this treasure-trove of a volume. These letters from Bose to Einstein may be viewed online at Albert Einstein Archives (https://albert-einstein.huji.ac.il). The reader may also profit by reading *The Making of Modern Physics in Colonial India* by S. Banerjee, Routledge, 2020. In this book the lives and works of Bose, C. V. Raman, and M. Saha are placed in the colonial context.

Einstein's translation of Bose's paper appeared in volume 26 (pp. 168-171) of *Zeitschrift für Physik* (Journal of Physics) in 1924 under the title *Plancks Gesetz und Lichtquantenhypothese* (Planck's Law and the Light-Quantum Hypothesis). "The Indian Bose has given a beautiful derivation of Planck's law including the constant $8\pi\nu^2/c^3$", wrote Einstein to Ehrenfest on July 12.

We now return to the first factor $8\pi\nu^2/c^3$, where, according to Einstein (see his response to Bose on 2 July, 1924), Bose's derivation is not fully quantum-theoretical. The reason for this is the need to multiply the factor $4\pi\nu^2/c^3$, which emerged from quantum-theoretical considerations, by a factor 2. In the paper published in *Zeitschrift für Physik* in German by Einstein upon Bose's request, we read (in English translation), "It seems, however, appropriate to multiply this number $[4\pi\nu^2/c^3]$ once again by 2 in order to take into account the fact of polarization [...]." Therefore, in the published paper, it appears as if Bose is introducing an additional factor of 2 accounting for the two polarization states of the electromagnetic field, which is a classical concept on top of the quantum-theoretically motivated factor $4\pi\nu^2/c^3$, thereby not making the entire argument quantum-based. However, the following colorful story appears in Partha Ghose's recollections[6]:

I spoke with Bose regarding the factor of 2 in his derivation. I had gone to meet him one afternoon and found him in an introspective mood. He was reminiscing about his meetings and conversations with Einstein in Berlin during 1925. Suddenly he said to me that he would like to tell me something confidential that I must never divulge to anyone. He got up, closed the doors, and shut the windows. I was in great suspense. He sat down and told me the following story.

"You know," he said, "my deduction of the Planck law had a factor of 2 missing. So I proposed that it came from the fact that the photon had a spin, and that it can spin either parallel or antiparallel to its direction of motion. That would give the additional factor of 2. But the old man [meaning Einstein] crossed it out and said it was not necessary to talk about spin, the factor of 2 comes from the two states of polarization of light." Then he smiled mischievously and said to me rhetorically, "I can understand a spinning particle, but what is the meaning of the polarization of a particle?"

I was stunned! I immediately said to him, Sir, when photon spin was eventually discovered, why didn't you tell Einstein that you had discovered it already in 1924? No doubt a person like Einstein would have stood by you [...]. He looked at me calmly and said, "How does it matter who discovered it?" And then with a sense of triumph, he said, "It *has* been discovered, hasn't it!"

That was Bose.

Much later, around 1993 when we were preparing for the Bose centenary, while browsing old journals in the library of the Indian Association for the Cultivation of Science, I chanced upon a 1931 paper by Chandrasekhara Venkata Raman and Suri Bhagavantam in the *Indian Journal of Physics* [Vol. 6 (1931), 353] with the title, *Experimental Proof of the Spin of the Photon*. I was intrigued and started reading it. I was simply amazed by what it contained:

[6]See *Counting in the Dark* by P. Ghose, published in the online magazine, *Inference*, Vol. 7, No. 3 (October 2022).

> In his well-known derivation of the Planck radiation formula from quantum statistics, Prof. S. N. Bose obtained an expression for the number of cells in phase-space occupied by the radiation, and found himself obliged to multiply it by a numerical factor 2 in order to derive from it the correct number of possible arrangements of the quantum in unit volume. The paper as published did not contain a detailed discussion of the necessity for the introduction of this factor, but we understand from a personal communication by Prof. Bose that he envisaged the possibility of the quantum possessing besides energy $h\nu$ and linear momentum $h\nu/c$ also an intrinsic spin or angular momentum $\pm h/2\pi$ round an axis parallel to the direction of its motion. The weight factor 2 thus arises from the possibility of the spin of the quantum being either right-handed or left-handed, corresponding to the two alternative signs of the angular momentum. [...]

Their experiment conclusively showed that Bose's view was the correct one and provided the first experimental measurement of the photon spin. This fact is hardly known to scientists and historians of science.

I was beside myself with joy! But at the same time I was puzzled by the great secrecy with which Bose had confided the story. After all, Raman had already published it way back in 1931. Did Bose forget about Raman's paper? Surely that could not be the case, for how could anyone forget such a dramatic justification of one's revolutionary idea? And Bose had a prodigious memory. Was it that Raman did not inform Bose of his results? That also seems unlikely.

In any case, since then, I have had no compunction in breaking my pledge not to divulge the story. It is important for the history of science that it becomes widely known.

If this is the case, then Bose had introduced the notion of spin (intrinsic angular momentum) to the photon in his original English version of the paper, which he requested Einstein to translate. Thus, a photon with a given energy may have angular momentum pointing along or opposite to the direction of its motion, thereby introducing a multiplicative factor of 2, which doubles the available states for a photon thereby bringing the factor $4\pi\nu^2/c^3$ to align with the phase space factor $8\pi\nu^2/c^3$, and in the process, making the entire argument quantum-theoretical. To settle the mystery of what Bose's original article to Einstein in English contained, it seems the relevant historical documents have been lost: Bose apparently did not keep a copy of the paper he sent Einstein, and the Einstein archive does not contain the paper he received from Bose.

Whatever the truth regarding the origins of the "missing" factor of two may be, Einstein was able to see that Bose's ideas in dealing with massless electromagnetic quanta or photons can be extended to massive particles of similar nature (in today's parlance, obeying the same statistics). As a prelude to what may be anticipated, Einstein included the following note in his translation of Bose's paper: "In my opinion Bose's derivation signifies an important advance. The method used here gives the quantum theory of an ideal gas as I will work out elsewhere."

In fact, the indications are that it took only a week or so for Einstein to derive far reaching consequences by applying Bose's methods to ideal gases. These meditations by Einstein resulted in

two papers (both presented to the Prussian Academy of Sciences)[7]: *Quantentheorie des einatomigen idealen Gases* (On the Quantum Theory of the Monatomic Ideal Gas), which appeared in 1924, and *Quantentheorie des einatomigen idealen Gases. Zweite Abhandlung.* (On the Quantum Theory of the Monatomic Ideal Gas. Second Paper), which appeared in 1925. In the first paper, Einstein derived the expression for the expected number of particles in a given state, essentially, $n = 1/\{\exp[h\nu/(k_B T)] - 1\}$. This is the beginning of what today is called the Bose-Einstein distribution. It is interesting to note that concerning the phase space volume of a quantum state, Einstein borrows from Bose's assumption without proof that the said volume is h^3. Einstein writes:

"The path to be taken below, following Bose, is to be described thus: The phase space of an elementary structure (here of a monatomic molecule) is divided, with reference to a given (three-dimensional) volume, into 'cells' of extension h^3."

In the second paper, he arrived at the existence of what today is called Bose-Einstein condensation. In this second paper, Einstein states the phenomenon of condensation as follows: "What happens, though, if at [critical temperature] I now let the density of the substance increase even more (e.g., by isothermal compression)? I claim that, in this case, a steadily growing number of molecules compared to the total density will go over into the first quantum state (state without kinetic energy), while the remainder of the molecules will distribute themselves [according to the value of a parameter that determines the distribution]." Further on, Einstein states the condensation as a theorem: "According to the equation of state of an ideal gas derived here, a maximal density for molecules is in agitation at any temperature. Upon surpassing this density, the excess molecules drop out as immobile (they 'condense' without attractive forces). The remarkable thing is that the 'saturated ideal gas' not only represents the state of maximal density possible for moving gas molecules, but also represents the density at which the gas is at thermodynamic equilibrium with the 'condensate'. So, nothing analogous to 'supersaturated vapor' exists for the ideal gas."

3 The Bose-Einstein Distribution

Consider a system consisting of N number of identical bosons, which are indistinguishable. Indistinguishability means that the trajectories of two bosons cannot be distinguished in the phase space since, according to quantum mechanics, the uncertainty principle of Heisenberg puts a lower limit on the resolution of positions and momenta for these particles. (In contrast, according to classical mechanics, the particles are distinguishable since their positions and momenta can be resolved with infinite precision. Hence, particles that behave according to the statistical laws of Maxwell and Boltzmann are distinguishable.)

Let the N bosons be distributed in various energy levels (indexed by i) with each level having

[7] Both these papers in their English translation can be found in *The Essential Einstein: Scientific Writings*, D. K. Buchwald and T. Sauer (Eds.), Princeton University Press, 2025, pp. 346-368.

energy ϵ_i. Further, let there be g_i number of distinguishable states in each energy level i. Therefore, the degeneracy of the energy level i is g_i. If there are n_i number of bosons in the energy level i, how many distinct ways are there for the n_i bosons to be distributed among the g_i states corresponding to energy ϵ_i?

As an example, suppose $n_i = 3$ and $g_i = 2$ for some level i. So we are interested in determining the number of distinct ways in which 3 indistinguishable bosons can be put in 2 distinguishable states (each state corresponding to the same energy since they are degenerate). Since $n_i = 3$, let us first draw the 3 bosons as three circles, each of which is indistinguishable from the others since the bosons are indistinguishable.

• • •

We now need to distribute them in 2 states since $g_i = 2$. For this we can introduce 1 vertical line so that bosons to the left of this line belong to one (first) state and the bosons to the right of this line belong to the other (second) state. This vertical line can then be drawn at 4 different places as follows, thereby distributing the 3 bosons between the 2 states in 4 distinct ways:

| • • •
• | • •
• • | •
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The distribution in the first row indicates that there are no bosons in the first state; all 3 bosons are in the second (degenerate) state. The distribution in the second row indicates that there is 1 boson in the first state with the second (degenerate) state carrying 2 bosons; and so on.

Consider now tossing a coin 4 times and asking how many ways there are in obtaining only 1 Heads ($\mathbb{H}$), or alternatively, only 3 Tails ($\mathbb{T}$). We may obtain the single $\mathbb{H}$ on the first toss, or on the second toss, or on the third toss, or on the fourth toss. Alternatively, we may obtain the 3 $\mathbb{T}$s on the tosses $(2, 3, 4)$, or on the tosses $(1, 3, 4)$, or on the tosses $(1, 2, 4)$, or on the tosses $(1, 2, 3)$. These scenarios are represented below where the numbers within the parentheses correspond to the toss:

(1)	(2)	(3)	(4)
$\mathbb{H}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{H}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{H}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{H}$

We clearly can see a one-to-one correspondence between the distribution of 3 bosons in 2 states (which was achieved with 1 partition) and the tossing of a coin 4 times and listing the possible ways in which 3 $\mathbb{T}$s can be obtained. Another way of stating this is that given the 4 numbers from the set $\{1, 2, 3, 4\}$ how many ways are there in selecting 3 distinct numbers? The answer is

$$\frac{4!}{3!(4-3)!} = 4.$$

The resulting answer, 4, corresponds to the 4 rows listed in the coin toss example.

The correspondence we have established implies that we can consider a vertical line (that is, a state partition) as a $\mathbb{H}$ and a circle (that is, a boson) as a $\mathbb{T}$. Just as the bosons in the above example were partitioned by 1 vertical line to represent 2 states, a g_i number of states would require $g_i - 1$ vertical lines. Thus, the scenario where n_i indistinguishable bosons are distributed in g_i distinguishable states is akin to considering $g_i - 1$ $\mathbb{H}$s and n_i $\mathbb{T}$s when a coin is tossed $n_i + (g_i - 1)$ times. Therefore, the number of distinct ways in which the n_i indistinguishable bosons can be arranged in g_i distinguishable states is equivalent to finding the number of distinct ways in which n_i $\mathbb{T}$s are obtained in $n_i + (g_i - 1)$ coin tosses [or, equivalently, $g_i - 1$ $\mathbb{H}$s are obtained in $n_i + (g_i - 1)$ coin tosses], which is

$$\frac{(n_i + g_i - 1)!}{n_i![(n_i + g_i - 1) - n_i]!} = \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}.$$

Each distinct arrangement of bosons among the degenerate states corresponding to the energy level i is a microstate ($\mathcal{W}_i$). Therefore, the number of microstates in which n_i indistinguishable bosons can be distributed among g_i distinguishable states in energy level i is

$$\boxed{\mathcal{W}_i = \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}, \quad g_i \geq 1. \quad \textbf{number of microstates in energy level } i} \tag{1}$$

Note that for the above expression to be defined, $g_i \geq 1$; in other words, there must be at least one state in a given energy level i. This is not surprising since we cannot speak of stateless energy levels. Note also that if there are no bosons in a given energy level, then $n_i = 0$. Since $0! = 1$, this means that the number of microstates when $n_i = 0$ is one. Thus $W_i \neq 0$ for any energy level i. If

there are $\mathcal{W}_1$ microstates in energy level $i = 1$ and $\mathcal{W}_2$ microstates in energy level $i = 2$, and so on, then, since the number of microstates in one energy level is independent from the number of microstates in another energy level, the total number of microstates ($\mathcal{W}$) is given by the product of all the $\mathcal{W}_i$s:

$$\mathcal{W} = \prod_{i=1} \mathcal{W}_i = \prod_{i=1} \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}. \tag{2}$$

Since $W_i \neq 0$ for any given energy level, it follows that the total number of microstates, $\mathcal{W}$, will never be zero. Now, the energy ϵ_i and the degeneracy g_i of a level i is fixed for a given system while the number n_i occupying the states may change depending on the system conditions such as the temperature. Our goal now is to find the number n_i of bosons that would maximize the total number of microstates. The most likely macrostate of the system is the one with the most (total) number of microstates. To find this n_i we need to differentiate (2) with respect to n_i, set the resulting expression to zero, and solve for n_i. However, this maximization needs to be done subject to two constraints: (1) the total number of particles (N) in the system is constant, (2) the total energy of the system (E) is constant. These two constraints can thus be cast as:

$$\sum_i n_i = N = \text{constant}, \tag{3}$$

$$\sum_i \epsilon_i n_i = E = \text{constant}. \tag{4}$$

To simplify the computation, let us first assume that $g_i \gg 1$. In that case,

$$\mathcal{W} \simeq \prod_{i=1} \frac{(n_i + g_i)!}{n_i! g_i!}, \quad g_i \gg 1.$$

Since the logarithm of $\mathcal{W}$ is a monotonic function of $\mathcal{W}$, if $\mathcal{W}$ is maximized at a particular n_i, then so is the logarithm of $\mathcal{W}$. Therefore, taking the (natural) logarithm on both sides give

$$\ln \mathcal{W} \simeq \sum_i \{\ln(n_i + g_i)! - \ln n_i! - \ln g_i!\}.$$

If we assume n_i and g_i to be large, then from Stirling's approximation, which reads $\ln x! \simeq x \ln x - x$ for large x (see Appendix A for a derivation), we have

$$\begin{aligned} \ln \mathcal{W} &\simeq \sum_i \{(n_i + g_i) \ln(n_i + g_i) - (n_i + g_i) - [n_i \ln n_i - n_i] - [g_i \ln g_i - g_i]\}, \\ \therefore \ \ln \mathcal{W} &\simeq \sum_i \{(n_i + g_i) \ln(n_i + g_i) - n_i \ln n_i - g_i \ln g_i\}. \end{aligned}$$

Keeping g_i constant, applying the differential operator yields

$$d\ln\mathcal{W} = \sum_i \{\ln(n_i + g_i) - \ln n_i\} dn_i.$$

Applying the differential operator to the two constraints (3) and (4) gives

$$\sum_i dn_i = 0, \quad \sum_i \epsilon_i \cdot dn_i = 0.$$

Let us introduce the Lagrange multipliers a and b (valid over all energy levels) such that

$$\sum_i a \cdot dn_i = 0, \quad \sum_i b\epsilon_i \cdot dn_i = 0.$$

Since these expressions vanish, they can be introduced in the expression for $d\ln\mathcal{W}$ so that it remains unchanged. Thus,

$$d\ln\mathcal{W} = \sum_i \{\ln(n_i + g_i) - \ln n_i + a + b\epsilon_i\} dn_i.$$

Therefore,

$$\frac{d\ln\mathcal{W}}{dn_i} = \sum_i \{\ln(n_i + g_i) - \ln n_i + a + b\epsilon_i\}. \tag{5}$$

The n_i at which $\ln\mathcal{W}$ is maximized, and hence, $\mathcal{W}$ is maximized, follows by setting the right-hand side of the above expression to zero. Thus, $\sum_i \{\ln(n_i + g_i) - \ln n_i + a + b\epsilon_i\} = 0$ implies that

$$\ln(n_i + g_i) - \ln n_i + a + b\epsilon_i = 0 \quad \forall i. \tag{6}$$

The fact that this indeed results in a value for n_i that maximizes the total number of microstates can be verified by taking the derivative of (5) with respect to n_i again while keeping g_i, a, b, and ϵ_i constant. This gives

$$\frac{d^2\ln\mathcal{W}}{dn_i^2} = \sum_i \left\{ \frac{1}{n_i + g_i} - \frac{1}{n_i} \right\} = -\frac{g_i}{n_i(n_i + g_i)} < 0.$$

Thus, the second derivative of $\ln\mathcal{W}$ with respect to n_i is negative since n_i and g_i are finite and non-negative. Therefore, the total number of microstates is maximized for which n_i satisfies (6). We can now rearrange (6) to read

$$1 + \frac{g_i}{n_i} = e^{-a - b\epsilon_i}. \tag{7}$$

The left-hand side of (7) is always > 1 since g_i and n_i are both finite and non-negative. However,

the right-hand side may tend to 0 if $\epsilon_i \to \infty$ while a and b remain finite and positive. To avoid such a contradiction where $\epsilon_i \geq 0$, let us define $b < 0$ such that $b = -\beta$ where $\beta > 0$ has units of inverse energy. Thus,

$$1 + \frac{g_i}{n_i} = e^{-a+\beta\epsilon_i}, \;\; \beta > 0, \;\; \epsilon_i \geq 0.$$

Solving this last expression for n_i we obtain

$$n_i = \frac{g_i}{e^{\beta\epsilon_i - a} - 1}, \;\; \beta > 0, \;\; \epsilon_i \geq 0. \tag{8}$$

Since the number of bosons (n_i) and the number of states (g_i) in an energy level i are both finite and non-negative, it follows that

$$e^{\beta\epsilon_i - a} - 1 > 0 \implies e^{\beta\epsilon_i - a} > 1 \implies \beta\epsilon_i - a > \ln(1) \implies \beta\epsilon_i - a > 0 \implies \beta\epsilon_i > a \;\; \forall i.$$

Note that $\beta\epsilon_i > a$ must be valid for any energy level i since a is independent of the energy level (there is no energy level index i on a). Since the lowest energy that a level may have is $\epsilon_i = 0$, this implies that $0 > a$. Thus, for bosons, $a < 0$. Since a is associated with the constraint that the total number of bosons in the system is fixed, this implies that for the total number of bosons in the system to be conserved a must be negative. We have therefore derived the Bose-Einstein distribution, which gives the likely number of bosons in an energy level i when the total number of bosons, N, is conserved. With $\beta = 1/(k_B T)$, where k_B is the Boltzmann constant and T is the absolute temperature,

$$n_i = \frac{g_i}{e^{\beta\epsilon_i - a} - 1}, \;\; \beta = \frac{1}{k_B T} > 0, \;\; \epsilon_i \geq 0, \;\; a < 0. \;\; \textbf{Bose-Einstein distribution} \text{ for } N = \text{const.} \tag{9}$$

Note that it is possible for the denominator of the Bose-Einstein distribution to satisfy the condition $0 < e^{\beta\epsilon_i - a} - 1 < 1$, in which case $n_i/g_i > 1 \implies n_i > g_i$. That is, in a given energy level i it is possible for there to be more bosons than there are states. This then implies that more than one boson can be in a given state, therefore violating the Pauli Exclusion Principle.

The result (9), as we have stated, is valid when the total number of bosons, N, is conserved so that N remains constant. In that case, as we have determined, the Lagrange multiplier a is negative. It is possible that the total number of bosons in a system do not remain constant, in which case there is no constraint on N. This in turn implies that the Lagrange multiplier, a, must vanish when the number of bosons are not conserved. In this case, then, (9) reduces to

$$n_i = \frac{g_i}{e^{\beta\epsilon_i} - 1}, \;\; \beta = \frac{1}{k_B T} > 0, \;\; \epsilon_i \geq 0. \;\; \textbf{Bose-Einstein distribution} \text{ for } N \neq \text{const.} \tag{10}$$

Regardless of whether the total number of bosons are conserved, the total energy of the system

remains conserved. A further constraint can now be deduced. Let $e^{-a} = A$. Since any exponent of e is non-negative, it follows, from a purely mathematical standpoint, that $A \geq 0$. As we have seen, when the number of bosons are not conserved $a = 0$, which implies that $A = 1$. On the other hand, when the number of bosons are fixed, we have seen that $a < 0$, which means that in this context, the exponent $(-a) > 0$. But any exponent of e greater than zero is greater than one. It therefore follows that $A > 1$ when the number of bosons are conserved. Thus, from a physical standpoint, for bosons, $e^{-a} = A \geq 1$. In essence, $A \geq 1$ guarantees that the number of bosons, n_i, in an energy level i will never be negative. To see this, note that since $\beta\epsilon_i \geq 0$, $e^{\beta\epsilon_i} \geq 1$. Therefore, if $A \geq 1$, then $Ae^{\beta\epsilon_i} \geq 1$; so, $Ae^{\beta\epsilon_i} - 1 \geq 0$, making n_i non-negative since g_i is non-negative as well. We summarize these results below:

Bose-Einstein Distribution

Let there be N bosons in a system. If the energy level i has degeneracy g_i, that is, the number of states with energy $\epsilon_i \geq 0$ is $g_i \gg 1$, then the most likely number of bosons in energy level i is

$$n_i = \frac{g_i}{e^{\beta\epsilon_i - a} - 1}, \quad \beta = \frac{1}{k_B T} > 0, \tag{11}$$

where $a < 0$ if $\sum_i n_i = N$ is conserved, and $a = 0$ if N is not conserved. In all cases the total energy $E = \sum_i \epsilon_i n_i$ is conserved. If $e^{-a} = A$, then

$$n_i = \frac{g_i}{Ae^{\beta\epsilon_i} - 1}, \quad A \geq 1, \tag{12}$$

where $A = 1$ when the number of bosons is not conserved, and $A > 1$ when the number of bosons is conserved. Overall, the condition $A \geq 1$ guarantees that the number of bosons, n_i, in an energy level i, is non-negative. k_B is the Boltzmann constant and T is the absolute temperature.

4 The Phase Space Volume of a State

In his derivation of Planck's blackbody radiation formula using quantum statistics for photons, Bose assumed the six-dimensional phase space volume of a quantum state to be h^3, which has been introduced by Planck in the context of a massive oscillator. We provide below a derivation of the phase space volume of a quantum state using a semi-classical approach.[8]

Consider a classical harmonic oscillator of mass m in one-dimension. It's Hamiltonian is given by

$$H(p,q) = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 q^2,$$

where p is the momentum, q is the position, and ω is the angular frequency of oscillations. Quantizing this harmonic oscillator yields the quantized energy-levels given by

$$E_{\mathfrak{n}} = \left(\mathfrak{n} + \frac{1}{2}\right)\hbar\omega,$$

where $\mathfrak{n} = 0, 1, 2, \cdots$ is the quantum number and $\hbar = h/(2\pi)$, h being the Planck's constant (see Appendix B for a heuristic derivation). We here have both classical and quantum aspects describing the oscillator: the classical aspects are represented by the Hamiltonian while the quantum aspects are represented by the energy; therefore, our treatment of the oscillator is semi-classical.

Our goal now is to find the phase space volume of a quantum state of the harmonic oscillator. Since the oscillator is moving in one-dimension, this means its phase space is two-dimensional (one dimension to describe the positions in one-dimensional physical space, and the second dimension to describe its one-dimensional momentum). Since there is one quantum state between the quantum numbers $\mathfrak{n}$ and $\mathfrak{n}+1$, to find the phase space (ps) volume $V_{ps/qs}$ of a quantum state (qs), we need to perform the integration

$$V_{ps/qs} = \int_{\mathfrak{n}}^{\mathfrak{n}+1} dp \cdot dq.$$

Here, $dp \cdot dq$ represents the infinitesimal area of a rectangle of length dp and width dq in the two-dimensional phase space. In order to compute this integral let us first make the following variable changes:

$$\mathfrak{p}^2 = \frac{p^2}{m} \implies p = \sqrt{m}\mathfrak{p}, \quad \mathfrak{q}^2 = m\omega^2 q^2 \implies q = \mathfrak{q}/(\sqrt{m}\omega).$$

[8] In this sense we may posit that Bose's derivation of Planck's radiation law is not fully quantum-mechanical, but instead, is semi-classical though it does remove reference to electromagnetic waves.

Therefore,

$$dp = \sqrt{m}\, d\mathfrak{p}, \;\; dq = \frac{1}{\sqrt{m}\omega}\, d\mathfrak{q}.$$

Thus,

$$V_{ps/qs} = \frac{1}{\omega} \int_{\mathfrak{n}}^{\mathfrak{n}+1} d\mathfrak{p} \cdot d\mathfrak{q}.$$

Now $d\mathfrak{p} \cdot d\mathfrak{q}$ can be considered as the infinitesimal area of a rectangle in the two-dimensional cartesian plane. Since $H(p,q)$ does not depend on time explicitly, it is a constant of motion. Therefore, for a given $H(p,q)$, the trajectory of the oscillator in (p,q) space is elliptical. However, since $2H(\mathfrak{p},\mathfrak{q}) = \mathfrak{p}^2 + \mathfrak{q}^2$, this means that for a given $H(\mathfrak{p},\mathfrak{q})$, the trajectory of the oscillator in $(\mathfrak{p},\mathfrak{q})$ space is circular with radius $r = \sqrt{2H(\mathfrak{p},\mathfrak{q})}$. Utilizing polar coordinates (r,θ) on the $(\mathfrak{p},\mathfrak{q})$ plane, so that $\mathfrak{p} = r\cos\theta$ and $\mathfrak{q} = r\sin\theta$, we have $d\mathfrak{p} \cdot d\mathfrak{q} = rd\theta \cdot dr = rdr \cdot d\theta$. Since the trajectory in the $(\mathfrak{p},\mathfrak{q})$ plane is circular, θ ranges from 0 to 2π. The range of r can be taken from $r_{\mathfrak{n}}$ to $r_{\mathfrak{n}+1}$ where the former corresponds to the circle with quantum number $\mathfrak{n}$ and the latter corresponds to the circle with quantum number $\mathfrak{n}+1$. Therefore,

$$V_{ps/qs} = \frac{1}{\omega} \int_{\mathfrak{n}}^{\mathfrak{n}+1} rdr \cdot \int_0^{2\pi} d\theta = \frac{2\pi}{\omega} \int_{\mathfrak{n}}^{\mathfrak{n}+1} rdr = \frac{\pi}{\omega}\left(r_{\mathfrak{n}+1}^2 - r_{\mathfrak{n}}^2\right).$$

Since $\mathfrak{p}^2 + \mathfrak{q}^2 = 2H(\mathfrak{p},\mathfrak{q}) = 2H = r^2$, this implies that $r_{\mathfrak{n}}^2 = 2H_{\mathfrak{n}}$. Since $H_{\mathfrak{n}}$ implies that the Hamiltonian is quantized, we can substitute $E_{\mathfrak{n}}$ in place of $H_{\mathfrak{n}}$. Therefore, $r_{\mathfrak{n}}^2 = 2E_{\mathfrak{n}}$ and $r_{\mathfrak{n}+1}^2 = 2E_{\mathfrak{n}+1}$. Thus, the two-dimensional phase space volume of a quantum state of the harmonic oscillator is

$$V_{ps/qs} = \frac{2\pi}{\omega}\left(E_{\mathfrak{n}+1} - E_{\mathfrak{n}}\right) = \frac{2\pi}{\omega}\left[\left(\mathfrak{n}+1+\frac{1}{2}\right) - \left(\mathfrak{n}+\frac{1}{2}\right)\right]\hbar\omega = h.$$

This result shows that the phase space volume of a quantum state is independent of the properties such as the mass and the angular frequency of the oscillator. Therefore, the result can be generalized to any particle. We have thus established that the two-dimensional phase space volume of a quantum state for a particle moving in one spatial dimension is h. Therefore, the six-dimensional phase space volume of a quantum state for a particle moving in three spatial dimensions is $h \times h \times h = h^3$. We can summarize the result as follows:

Phase Space Volume of a Quantum State

A particle moving in $\mathtt{d}$ spatial dimensions has a phase space dimension of $2\mathtt{d}$. The phase space (ps) volume of a quantum state (qs) of the particle is then given by

$$V_{ps/qs} = h^{\mathtt{d}}, \tag{13}$$

where h is the Planck's constant.

5 Density of States

So far in our discussion the number of states, g_i, in an energy level i with energy ϵ_i has only been an abstract entity devoid of any structure. That is, in a system, we expect the number of accessible states to depend someway on the energy of the particles. Our goal now is to find an expression for the number of states. Toward this end, we again have to rely on a semi-classical approach where our starting point is the volume of the system in phase space.

Consider a system of particles in three-dimensional space so that $\mathtt{d} = 3$. From (13) it follows, then, that the phase space volume of a quantum state of this system is $V_{ps/qs} = h^3$. If there are N_s number of states in total in the system, then the total phase space volume (V_{ps}) of the system is

$$V_{ps} = V_{ps/qs} N_s = h^3 N_s. \tag{14}$$

This is the result for V_{ps} from the quantum side of the semi-classical approach (although we note that the derivation of the volume of a quantum state itself was semi-classical). Now, from the classical side of the semi-classical approach,

$$V_{ps} = \int dV_{ps} = \int_{\text{space}} dx \cdot dy \cdot dz \int_{\text{momentum}} dp_x \cdot dp_y \cdot dp_z.$$

If the (constant) spatial volume of the system is V, then

$$\int_{\text{space}} dx \cdot dy \cdot dz = V.$$

Since the momentum-space is also three-dimensional with components (p_x, p_y, p_z), we can consider a sphere in this momentum-space with radius $p = (p_x^2 + p_y^2 + p_z^2)^{1/2}$ with infinitesimal thickness dp. The volume of this momentum-space shell is then

$$\int_{\text{momentum}} dp_x \cdot dp_y \cdot dp_z = 4\pi p^2 \, dp.$$

Therefore,

$$V_{ps} = V \int 4\pi p^2 \, dp = \int 4\pi V p^2 \, dp. \tag{15}$$

We can now equate (14) and (15) to obtain

$$N_s = \int \frac{4\pi V}{h^3} p^2 \, dp. \quad \text{number of particle states} \tag{16}$$

Now, if the density of particle states is $g(p)$, which is the number of particle states per unit momentum range, then the number (dN_s) of states between p and $p + dp$ must be $dN_s = g(p) \, dp$. Therefore,

$$N_s = \int dN_s = \int g(p) \, dp = \int \frac{4\pi V}{h^3} p^2 \, dp.$$

It then follows by comparison that

$$\boxed{g(p) = \frac{4\pi V}{h^3} p^2. \quad \text{density of particle states; momentum representation}} \tag{17}$$

Thus, the density of particle states, when written or represented in terms of momentum, scales as $g(p) \sim p^2$. Note that the above expression is independent of the mass of the particles since no such consideration was entered into its proof. Therefore, equation (17) is a fundamental expression from which we can branch out to either massive or massless particles. Note also that in deriving the expression (17), nothing explicit about bosons was considered. Therefore, it applies to any particle type in general.

If we consider a system of particles, all of mass $m \neq 0$, to be ideal, that is, if there are no interactions among the particles, then the energy (E) of a particle with momentum p is given by $\epsilon = p^2/(2m)$, where all energy takes the form of classical kinetic energy. We may then think that the density of particle states, when written or represented in terms of energy, scales as $g(\epsilon) \sim \epsilon$. However, this is incorrect. To see this, note that $p = \sqrt{2m\epsilon}$. Therefore, $dp = m \, d\epsilon / (\sqrt{2m\epsilon})$. Now, if the density of states, written in terms of energy, is $g(\epsilon)$ – that is, $g(\epsilon)$ is the number of particle states per unit energy range – then the number (dN_s) of states between ϵ and $\epsilon + d\epsilon$ must be $dN_s = g(\epsilon) \, d\epsilon$. Therefore,

$$N_s = \int dN_s = \int g(\epsilon) \, d\epsilon.$$

Thus,

$$N_s = \int g(\epsilon)\, d\epsilon = \int g(p)\, dp = \int \frac{4\pi V}{h^3} p^2\, dp = \int \frac{4\pi V}{h^3}(2m\epsilon)\frac{m}{\sqrt{2m\epsilon}}\, d\epsilon = \int \frac{2\pi V}{h^3}(2m)^{3/2}\epsilon^{1/2}\, d\epsilon.$$

It then follows from comparison that

$$\boxed{g(\epsilon) = \frac{2\pi V}{h^3}(2m)^{3/2}\epsilon^{1/2}, \quad m \neq 0. \quad \text{density of particle states; energy representation}} \tag{18}$$

Thus, the density of particle states, when written or represented in terms of energy, scales as $g(\epsilon) \sim \sqrt{\epsilon}$. For the ensuing derivations let us rearrange (18) with the use of $\hbar = h/(2\pi)$ to read

$$\boxed{g(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}, \quad m \neq 0. \quad \text{density of particle states; energy representation}} \tag{19}$$

It is important to recognize that when we refer to "density" in the present context, it refers to a unit range (of momentum, energy, frequency, etc.) and not to a unit volume of space. In Section 9 we use (17) to derive the density of states in terms of energy of massless photons, which then directly leads us to Planck's blackbody radiation law.

6 Bose-Einstein Condensation

Consider the number of bosons in an energy level i as given in (12):

$$n_i = \frac{g_i}{Ae^{\beta\epsilon_i} - 1}, \quad A \geq 1.$$

The index i refers to discrete energy states. Since we are utilizing a semi-classical approach, let us assume that the energy states are continuous and remove the index so that $n_i \to n(\epsilon)$, $g_i \to g(\epsilon)$, and $\epsilon_i \to \epsilon$. Therefore, the number of bosons with energy ϵ per unit energy range is given by

$$n(\epsilon) = \frac{g(\epsilon)}{Ae^{\beta\epsilon} - 1}, \quad g(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}, \quad m \neq 0, \quad A \geq 1. \tag{20}$$

A little reflection is needed at this juncture. Since $g(\epsilon) \sim \sqrt{\epsilon}$, this implies that as $\epsilon \to 0$, $g(\epsilon) \to 0$. This in turn implies that as $\epsilon \to 0$, $n(\epsilon) \to 0$ (for $A > 1$). Since $\epsilon = 0$ indicates ground state energy, this means that the formalism we have so far indicates that there will be no ground state energy levels, and therefore, there will be no bosons in the ground state when their energy vanishes. So, if the bosons cannot occupy the ground state even when $\epsilon = 0$ since there are no

ground states, then we have to interpret (20) as giving us the number of bosons in excited states, which we will denote as $n^\bullet$. Thus,

$$\boxed{n^\bullet(\epsilon) = \frac{g(\epsilon)}{Ae^{\beta\epsilon} - 1}, \;\; g(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}, \;\; m \neq 0, \;\; A \geq 1.} \tag{21}$$

Therefore, we must obtain the total number of bosons in the excited states, $N^\bullet$, within the volume V by integrating $n^\bullet(\epsilon)$ over the entire energy range $\epsilon \geq 0$. (Note that although the lower bound of this integration is $\epsilon = 0$, it does not contain any bosons in the ground state according to the preceding discussion. Therefore, the integral captures all the bosons in the excited states.) Thus,

$$\boxed{\int_0^\infty n^\bullet(\epsilon)\, d\epsilon = N^\bullet \implies \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\sqrt{\epsilon}}{Ae^{\beta\epsilon} - 1}d\epsilon = N^\bullet, \;\; m \neq 0, \;\; A \geq 1.} \tag{22}$$

To evaluate this integral, let $u = \beta\epsilon$. Then,

$$\frac{V}{4\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\sqrt{u}}{Ae^u - 1}du = N^\bullet, \;\; m \neq 0, \;\; A \geq 1. \tag{23}$$

Let us define a function $\mathcal{R}(A)$ such that

$$\mathcal{R}(A) = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\sqrt{u}}{Ae^u - 1}du, \;\; A \geq 1. \tag{24}$$

Then (23) takes the form

$$\boxed{V\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}\mathcal{R}(A) = N^\bullet, \;\; m \neq 0, \;\; A \geq 1.} \tag{25}$$

Now, the Riemann zeta function, $\zeta_\mathfrak{s}$, is defined by the integral

$$\zeta_\mathfrak{s} = \frac{1}{\Gamma_\mathfrak{s}}\int_0^\infty \frac{u^{\mathfrak{s}-1}}{e^u - 1}du, \quad \text{Riemann zeta function}$$

where $\Gamma_\mathfrak{s}$ is the Gamma function defined by

$$\Gamma_\mathfrak{s} = \int_0^\infty \frac{u^{\mathfrak{s}-1}}{e^u}\, du, \;\; \Gamma_{\mathfrak{s}+1} = \mathfrak{s}\Gamma_\mathfrak{s}. \quad \text{Gamma function}$$

Therefore, when $A = 1$, it follows from (24) and the above two definitions, that

$$\int_0^\infty \frac{\sqrt{u}}{e^u - 1}du = \mathcal{R}(1)\frac{\sqrt{\pi}}{2} = \zeta_{3/2}\Gamma_{3/2}.$$

But $\Gamma_{3/2} = \sqrt{\pi}/2$; therefore,

$$\mathcal{R}(1) = \zeta_{3/2} \simeq 2.612.$$

Since A is in the denominator of (24), it follows that as A decreases $\mathcal{R}(A)$ must increase. Since the minimum value that A can have is 1, it then follows that the maximum of $\mathcal{R}(A)$ occurs at $A = 1$. Thus,

$$\mathcal{R}(A)|_{\max} = \mathcal{R}(1) \simeq 2.612.$$

Moreover, since u is non-negative (since ϵ is non-negative), and therefore $e^u \geq 1$, as A becomes much larger than unity, the product Ae^u becomes much larger than unity as well; therefore,

$$Ae^u - 1 \simeq Ae^u, \quad A \gg 1.$$

Hence, for large A,

$$\mathcal{R}(A) \simeq \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{u}}{Ae^u} du = \frac{2}{\sqrt{\pi}} \frac{1}{A} \int_0^\infty \frac{\sqrt{u}}{e^u} du = \frac{2}{\sqrt{\pi}} \frac{1}{A} \Gamma_{3/2} = \frac{2}{\sqrt{\pi}} \frac{1}{A} \frac{\sqrt{\pi}}{2} = \frac{1}{A}, \quad A \gg 1.$$

Therefore,

$$\mathcal{R}(A) \simeq \frac{1}{A}, \quad A \gg 1.$$

Substituting this in (25), we can find an expression for A when A is large:

$$A \simeq \frac{V}{N^\bullet} \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2}, \quad m \neq 0, \quad A \gg 1 \leftarrow \text{for high } T. \tag{26}$$

Let us now reflect on the expression given in (25), which is

$$V \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} \mathcal{R}(A) = N^\bullet, \quad m \neq 0, \quad A \geq 1. \tag{27}$$

At high temperatures where $A \gg 1$, we can posit that all the bosons contained in the volume V are in excited states. Therefore, since the total number of bosons is N, we posit that $N^\bullet = N$ at high temperatures. Also, as we have deduced above, $\mathcal{R}(A) \simeq 1/A$ at high temperatures. Thus,

$$V \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} \mathcal{R}(A) = N \implies A \simeq \frac{V}{N} \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2}, \quad m \neq 0, \quad A \gg 1 \leftarrow \text{for high } T. \tag{28}$$

Let us now consider the expression (27) when we start lowering the temperature. As T decreases, $\mathcal{R}(A)$ increases to keep the number of bosons in the excited states $N^\bullet$ conserved at the total number N. However, as we have seen, there is a limit to which $\mathcal{R}(A)$ can increase since $\mathcal{R}(A)|_{\max} = \mathcal{R}(1) = \zeta_{3/2} \simeq 2.612$. Therefore, a critical temperature, T_c, will be reached at which point $\mathcal{R}(A)$ is maximized at $\zeta_{3/2}$. Letting $T = T_c$, $\mathcal{R}(A) = \mathcal{R}(A)|_{\max} = \mathcal{R}(1) = \zeta_{3/2}$ (at which point, $A = 1$), and $N^\bullet = N$ in (27), it then follows that

$$\boxed{T_c = \frac{2\pi\hbar^2}{mk_B}\left(\frac{N}{V\zeta_{3/2}}\right)^{2/3}, \quad m \neq 0, \quad A = 1. \ \textbf{BEC critical temperature}} \tag{29}$$

This is the critical temperature at which a Bose-Einstein Condensate (BEC) would form; it depends on the number (N) of atoms, the volume (V) they occupy, and their mass (m). To understand the nature of this condensation, let us consider what would happen if we keep lowering the temperature below T_c. Since $\mathcal{R}(A)$ has reached its maximum at $T = T_c$ it cannot change any further. Therefore, as T continues to dip below T_c, the left-hand side of (27) will continue to decrease; this means that the right-hand side of (27), which is the number of bosons in the excited states will also continue to decrease. But how can the number of excited bosons decrease? The inescapable answer is, that for $T < T_c$, the excited bosons will start to occupy ground state energy levels. Since bosons are not restricted in their occupation number of energy levels, this means large numbers of bosons can occupy the ground state energy levels when $T < T_c$ thereby forming a condensate, which gives rise to the phenomenon known as the Bose-Einstein condensation.

Note that the condition $A \geq 1$ given in (25) makes sense when we interpret the number $N^\bullet$ on the right-hand side of that expression as the number of bosons in excited states. When $T > T_c$ all of the N bosons confined in the volume V can be considered to be in excited states. Therefore, when $T > T_c$, $N^\bullet = N$ is conserved since N is conserved, and therefore, when $T > T_c$, $A > 1$. When $T < T_c$, $N^\bullet$ is no longer conserved in the sense that bosons in excited states move to ground states; non-conservation of $N^\bullet$ implies that $A = 1$. The reader may note that if we have interpreted the right-hand side of (25) as N, which is always conserved, then we would be constrained to have $A > 1$ for all temperatures, which would have lead to inconsistencies in the formalism. Tracing back, it can be realized that an integral of the form $\int_0^\infty n(\epsilon)\, d\epsilon = N$ would have given rise to inconsistencies so that if $A > 1$ for all temperatures (since N is always conserved), then BEC cannot exist. For BEC to exist A must reach a value of 1, which in turn would mean N is not conserved, which also is unacceptable. We need $A \geq 1$ to cover all temperature ranges and for the BEC to form, in which case, we have to consider the integral $\int_0^\infty n^\bullet(\epsilon)\, d\epsilon$ and set it equal to $N^\bullet$, the number of bosons in the excited states, which is conserved at N for $T > T_c$ and not conserved for $T < T_c$. The critical temperature at which the BEC forms presents us with a special case where $T = T_c$; here the system starts to transition from the conserved state of $N^\bullet = N$ to the non-conserved state of $N^\bullet$ at which point A has just reached its minimum value of 1.

Returning to the computations, let us form $T_c^{3/2}$ from (29), which, upon multiplication with A given in (28), results in

$$A = \frac{1}{\zeta_{3/2}} \left(\frac{T}{T_c} \right)^{3/2}. \quad T \gg T_c \tag{30}$$

Let us now consider (25) when $T < T_c$. Since the number of excited bosons is not conserved when $T < T_c$, and therefore, $A = 1$, in this case (25) takes the form

$$V \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} \mathcal{R}(1) = N^{\bullet}|_{T<T_c}.$$

But $\mathcal{R}(1) = \zeta_{3/2}$. Therefore,

$$V \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} \zeta_{3/2} = N^{\bullet}|_{T<T_c}.$$

Multiplying this with $T_c^{3/2}$ formed from (29), it follows that

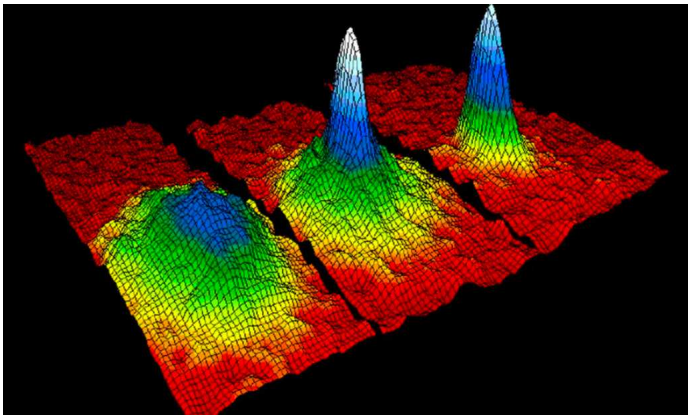
$$N^{\bullet}|_{T<T_c} = N \left(\frac{T}{T_c} \right)^{3/2}.$$

If $N_{\bullet}|_{T<T_c}$ is the number of bosons in the ground state when $T < T_c$, then, since $N = N_{\bullet}|_{T<T_c} + N^{\bullet}|_{T<T_c}$, it follows that

$$\boxed{\frac{N_{\bullet}|_{T<T_c}}{N} = 1 - \left(\frac{T}{T_c} \right)^{3/2}, \quad T_c = \frac{2\pi\hbar^2}{mk_B} \left(\frac{N}{V\zeta_{3/2}} \right)^{2/3}, \quad m \neq 0. \quad \textbf{condensate fraction}} \tag{31}$$

The quantity $N_{\bullet}|_{T<T_c}/N$ is known as the condensate fraction, which is the ratio of the number of bosons in the ground state to that of the total number of bosons in the system when $T < T_c$. As $T \to 0$, $N_{\bullet}|_{T<T_c} \to N$ as we expect all bosons to occupy the ground state at absolute zero. The experimental verification of Bose-Einstein condensation was achieved seven decades after its prediction by Einstein (see the image on the next page).[9] These condensates are typically $10 - 100$ μm across (about the width of a human hair) and is achieved at temperatures below 200 nK.

[9] *Bose-Einstein Condensation in a Gas of Sodium Atoms*, K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969, 1995; *Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor*, M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science, **269**, 198, 1995.



The momentum distribution profile of ^{87}Rb atoms as they transition to form a Bose-Einstein condensate in an atom trap. The left-most figure is above the critical condensate temperature where atoms have a relatively high thermal energy profile characterized by the broader momentum distribution. As the atoms begin to occupy the ground states, and thereby start to form the condensate, their momentum become confined to smaller and smaller ranges as can be seen by the narrower momentum distributions in the middle and the right-most figures. Image: NIST/JILA/CU-Boulder.

7 Some Properties of the Bose Gas

7.1 Total Energy

Using the expressions obtained above, we can compute the total energy of the bose gas. Since the number of bosons in energy level ϵ is given by $n^\bullet(\epsilon)$, the total energy, $\mathcal{E}$, can be found using the integral

$$\mathcal{E} = \int_0^\infty n^\bullet(\epsilon)\epsilon \, d\epsilon. \tag{32}$$

Substituting the expressions in (21) then gives us the integral

$$\mathcal{E} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\epsilon^{3/2}}{Ae^{\beta\epsilon} - 1} \, d\epsilon. \tag{33}$$

Using the substitution $u = \beta\epsilon$ and the expression for the critical temperature in (29), the above integral takes the form

$$\mathcal{E} = \frac{2}{\sqrt{\pi}}\frac{1}{\zeta_{3/2}} N k_B T \left(\frac{T}{T_c}\right)^{3/2} \int_0^\infty \frac{u^{3/2}}{Ae^u - 1} \, du. \tag{34}$$

Two cases can now be considered: At high temperatures, $A \gg 1$, and therefore $Ae^u - 1 \simeq Ae^u$.

The above integral then approximates to

$$\mathcal{E}|_{T\gg T_c} \simeq \frac{2}{\sqrt{\pi}}\frac{1}{\zeta_{3/2}} N k_B T \left(\frac{T}{T_c}\right)^{3/2} \frac{1}{A}\int_0^\infty \frac{u^{3/2}}{e^u}\, du, \quad T \gg T_c, \quad A \gg 1.$$

Using the expression in (30) and the fact that the integral results in $\Gamma_{5/2} = \frac{3}{2}\Gamma_{3/2} = \frac{3}{2}\frac{\sqrt{\pi}}{2}$, the total energy of the bosons at high temperature is

$$\mathcal{E}|_{T\gg T_c} \simeq \frac{3}{2} N k_B T \implies \mathcal{E}|_{T\gg T_c} \propto T. \tag{35}$$

At temperatures below the critical temperature, we have $A = 1$. In this case, then, (34) becomes

$$\mathcal{E}|_{T<T_c} = \frac{2}{\sqrt{\pi}}\frac{1}{\zeta_{3/2}} N k_B T \left(\frac{T}{T_c}\right)^{3/2} \int_0^\infty \frac{u^{3/2}}{e^u - 1}\, du. \tag{36}$$

The integral in this case evaluates to $\zeta_{5/2}\Gamma_{5/2}$. Thus,

$$\mathcal{E}|_{T<T_c} = \frac{3}{2}\frac{\zeta_{5/2}}{\zeta_{3/2}} N k_B T \left(\frac{T}{T_c}\right)^{3/2} \implies \mathcal{E}|_{T<T_c} \propto T^{5/2}. \tag{37}$$

Since $\mathcal{E}|_{T<T_c} \propto T^{5/2}$, the total energy of the bose gas is much more sensitive to temperature changes when $T < T_c$ than when $T \gg T_c$, where, in the latter case, $\mathcal{E}|_{T\gg T_c} \propto T$. We also note that the expression (35) agrees with the energy of an ideal gas under the Maxwell-Boltzmann distribution, as it should be. Further, the expression (35) agrees with what we would expect for a classical ideal gas where the equipartition of energy grants a $\frac{1}{2}k_B T$ amount of energy for each degree of freedom for a particle. Since bosons are free to move in three-dimensional space, this grants three degrees of freedom for each boson, thereby providing it with $\frac{3}{2}k_B T$ amount of energy. Thus, a gas consisting of N such bosons will have energy $\frac{3}{2}Nk_B T$ at high temperature.

7.2 Heat Capacity

Having computed the energy of the bose gas in the two temperature regimes, we can compute the heat capacity (C_V) of the gas at constant volume via

$$C_V|_{T<T_c} = \left(\frac{\partial \mathcal{E}|_{T<T_c}}{\partial T}\right)_V \propto T^{3/2}, \quad C_V|_{T\gg T_c} = \left(\frac{\partial \mathcal{E}|_{T\gg T_c}}{\partial T}\right)_V = \frac{3}{2}Nk_B.$$

Thus, below the critical temperature, the specific heat varies as $T^{3/2}$ while at temperatures much higher than that of the critical temperature, the specific heat is independent of the temperature.

7.3 Entropy

These results can now be used to compute the entropy (S) of the bose gas via

$$S|_{T<T_c} = \int_0^T \frac{C_V|_{T<T_c}}{T}\, dT \propto T^{3/2}, \quad S|_{T\gg T_c} = \int_{T_c}^T \frac{C_V|_{T\gg T_c}}{T}\, dT \propto \ln(T) - \ln(T_c).$$

As expected, when $T \to 0$, $S \to 0$ since all the bosons will occupy the ground state at absolute zero. The entropy scales logarithmically with high enough temperatures.

7.4 Pressure

The pressure of the ideal bose gas is given by

$$P = \frac{2}{3}\frac{\mathcal{E}}{V}. \tag{38}$$

Since $\mathcal{E}|_{T\gg T_c} \simeq \frac{3}{2}Nk_BT$, it follows that

$$P|_{T\gg T_c} \simeq Nk_BT/V \implies PV = Nk_BT, \quad T \gg T_c. \quad \text{classical ideal gas law} \tag{39}$$

Thus, at high enough temperatures, the pressure of the bose gas is proportional to the number of atoms and the temperature, and is inversely proportional to its volume. On the other hand, since $\mathcal{E}|_{T<T_c} = \frac{3}{2}\frac{\zeta_{5/2}}{\zeta_{3/2}}Nk_BT\left(\frac{T}{T_c}\right)^{3/2}$, and that $T_c = \frac{2\pi\hbar^2}{mk_B}\left(\frac{N}{V\zeta_{3/2}}\right)^{2/3}$, it follows that

$$P|_{T<T_c} = \zeta_{5/2}\left(\frac{m}{2\pi\hbar^2}\right)^{3/2}(k_BT)^{5/2}. \tag{40}$$

Hence, below the critical temperature, the pressure of the bose gas is independent of the number of atoms and its volume, and is proportional to $T^{5/2}$; the pressure also depends on the mass of the atoms and varies as $m^{3/2}$. Since $P|_{T<T_c}$ does not depend on the volume of the bose gas,

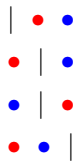
$$\left.\frac{\partial P}{\partial V}\right|_{T<T_c} = 0 \implies \left.\frac{\partial V}{\partial P}\right|_{T<T_c} \to \infty.$$

This indicates that below the critical temperature, the compressibility of the bose gas, defined as $\kappa = -\frac{1}{V}\frac{\partial V}{\partial P}$, is infinite. Compressibility indicates how much the gas shrinks when compressional forces are applied to it.

8 The Loss of Statistical Independence and the Rise of Quantum Entanglement

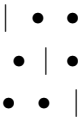
The Bose-Einstein statistics where more than one indistinguishable particle can occupy a given distinguishable state implies the loss of statistical independence. To see this, consider tossing two unbiased coins simultaneously. The two coins are distinguishable and each has a probability of $1/2$ of landing $\mathbb{H}$ or $\mathbb{T}$, which are two distinguishable states. The probability that both coins showing heads ($\mathbb{H}\mathbb{H}$) is then $(1/2) \times (1/2) = 1/4$. We also obtain a probability of $1/4$ for obtaining two heads if we enumerate the sample space of tossing two coins simultaneously. The sample space is $\{\mathbb{H}\mathbb{H}, \mathbb{H}\mathbb{T}, \mathbb{T}\mathbb{H}, \mathbb{T}\mathbb{T}\}$, yielding only one occurrence of $\mathbb{H}\mathbb{H}$ out of the four possible outcomes. This equivalency holds since the two coins are statistically independent and one coin cannot influence the behavior of the other. [Statistical independence of the two coins is encoded in the multiplication sign $\times$ in $(1/2) \times (1/2)$.]

By analogy, consider two distinguishable particles (blue and red) each of which can be in two possible distinguishable states with more than one particle allowed in a given state. The possible configurations are shown below:



There are four possible configurations with only one configuration where both particles belong to either the left or the right state. Therefore, the probability that both particles belong to either the left or the right state is $1/4 = (1/2) \times (1/2)$, which follows from the fact that each particle has a probability of $1/2$ in belonging to either the left or the right state and the fact that each particle behaves independently of the other. The probability that both particles belong in a single state is $(1/4) + (1/4) = 1/2$. Thus, the above configuration where particles are distinguishable with more than one particle allowed in a given state shows statistical independence. Such a configuration gives rise to Maxwell-Boltzmann statistics.

If we remove the criterion of distinguishability for the two particles while allowing for more than one particle to occupy a given (distinguishable) state, then the possible configurations are:



Now there are only three possible configurations. Although each particle still has a probability of $1/2$ in belonging to either the left or the right state, the probability that both particles belong

to either the left or the right state is $1/3$. This is in contrast to the probability of $1/4$ found if the particles are distinguishable. Since $1/3 \neq (1/2) \times (1/2)$, we see that statistical independence no longer holds for this configuration. Moreover, the probability that both particles belong in a single state is $(1/3) + (1/3) = 2/3$, which is different from the probability of $1/2$ found if the particles are distinguishable. We note that $2/3 > 1/2$, and therefore, when the particles are indistinguishable and more than one particle is allowed to occupy a given state, then the propensity for them to aggregate in a given state is more than when they are distinguishable. Thus, the above configuration where particles are indistinguishable with more than one particle allowed in a given state shows statistical dependence. Such a configuration gives rise to Bose-Einstein statistics.

An opposite configuration to Bose-Einstein statistics is also possible, where particles are indistinguishable but are subject to Pauli Exclusion Principle where no two particles can occupy a given state. The corresponding configuration in the context of the present example is then

$$\bullet \mid \bullet$$

Now there is only one possible configuration. Although each particle still has a probability of $1/2$ in belonging to either the left or the right state, the probability that both particles belong to either the left or the right state is 0. This is in contrast to the probability of $1/4$ found if the particles are distinguishable. Since $0 \neq (1/2) \times (1/2)$, we see that statistical independence no longer holds for this configuration. Moreover, the probability that both particles belong in a single state is $0 + 0 = 0$, which is different from the probability of $1/2$ found if the particles are distinguishable. We note that $0 < 1/2$, and therefore, when the particles are indistinguishable and no two particles can occupy a given state, then the propensity for them to aggregate in a given state is less than when they are distinguishable. Thus, the above configuration where particles are indistinguishable and are constrained by the Pauli Exclusion Principle also shows statistical dependence. Such a configuration gives rise to Fermi-Dirac statistics. Given the Bose-Einstein distribution (12),

$$n_i = \frac{g_i}{Ae^{\beta\epsilon_i} - 1}, \quad A \geq 1, \quad \text{Bose-Einstein distribution}$$

for Pauli Exclusion Principle to hold, there cannot be more particles in a given energy state than the corresponding degeneracy (that is $n_i \not> g_i$); otherwise, more than one particle will have to occupy a given (degenerate) energy state. Therefore, for Fermi-Dirac statistics it must always be true that $g_i \geq n_i$, which implies that $g_i/n_i \geq 1$. We note from the above Bose-Einstein distribution that this can always be satisfied if the negative sign in the denominator is turned to a positive sign; that is, if $\frac{g_i}{n_i} = Ae^{\beta\epsilon_i} + 1$, then

$$\frac{g_i}{n_i} = Ae^{\beta\epsilon_i} + 1 \geq 1.$$

Thus, the Fermi-Dirac distribution reads

$$\boxed{n_i = \frac{g_i}{Ae^{\beta\epsilon_i} + 1}, \quad A \geq 1. \quad \text{Fermi-Dirac distribution}} \tag{41}$$

The particles obeying the Fermi-Dirac distribution or statistics are called Fermions[10].

The loss of statistical independence, or, in other words, the statistical dependency of bosons disturbed Einstein. Referring to the new (Bose-Einstein) distribution, Einstein wrote in the sequel to his paper following Bose's work[11]:

> [A]ccording to Bose:
> A state is defined macroscopically by the indication of how many molecules are sitting inside each cell (complexion). the number of complexions for the νth infinitesimal region is then
>
> $$\frac{(n_\nu + z_\nu - 1)!}{n_\nu!(z_\nu - 1)!}.$$
>
> [In our notation $n_\nu = n_i$ and $z_\nu = g_i$.]
>
> [...] It is easily recognized that by this calculation approach the distribution of molecules over the cells is not treated as a statistically independent. This is because the cases [configurations], here called "complexions," would not be regarded of equal probability according to the hypothesis of an independent distribution of the individual molecules among the cells. For really statistically independent molecules, the counting of the "complexions" of differing probabilities would not yield the entropy correctly. *Consequently, the formula indirectly expresses a certain hypothesis about an initially completely puzzling mutual influence of the molecules that determines just the same statistical probability of the cases that are defined here as "complexions."* [My italics.]

Historically, this seems to be a starting point of Einstein's view that something "spooky" is going on in the quantum world leading to his thoughts about quantum entanglement and local realism (that is, observer-independent deterministic theories that prohibits influences propagating faster than the speed of light).

[10] The word *Fermion* was first coined by P. Dirac in 1947. These are particles with half-integer spin or intrinsic angular momentum.

[11] *Quantentheorie des einatomigen idealen Gases. Zweite Abhandlung.* (On the Quantum Theory of the Monatomic Ideal Gas. Second Paper) in *The Essential Einstein: Scientific Writings*, D. K. Buchwald and T. Sauer (Eds.), Princeton University Press, 2025, pp. 357-358.

9 Derivation of Planck's Radiation Law

We begin with the general result (17), which gives the density of boson states in the momentum representation:

$$g(p) = \frac{4\pi V}{h^3} p^2.$$

This is the number of boson states per unit momentum range where the bosons do not have any additional spin states. For photons (bosons), however, each state has a multiplicity of two to account for the two spin states of a photon. Therefore, the total number of photon states (N_s) is given by

$$N_s = \int 2 \times g(p)\, dp.$$

If the density of states is represented in terms of energy, then the total number of photon states is given by

$$N_s = \int g(\epsilon)\, d\epsilon.$$

Now, Planck's radiation law describes the distribution or the energy spectrum of electromagnetic radiation in equilibrium with a blackbody at absolute temperature T. It is typically expressed in terms of frequency ν of the radiation. Since the electromagnetic radiation consists of massless photons for which the energy (ϵ) and momentum (p) are related by

$$\epsilon = pc,$$

where c is the speed of light in a vacuum, it then follows that

$$N_s = \int 2 \times g(p)\, dp = \int 2 \times \frac{4\pi V}{h^3} p^2\, dp = \int \frac{8\pi V}{h^3 c^3} \epsilon^2\, d\epsilon = \int g(\epsilon)\, d\epsilon.$$

Thus, represented in terms of energy, the density of states of the photons is given by

$$g(\epsilon) = \frac{8\pi V}{h^3 c^3} \epsilon^2. \text{ (number of photon states per unit energy range)}$$

Since photons are being absorbed and emitted by a blackbody, their number is not conserved. Thus, taking $A = 1$ in (12), the number of photons per unit energy range is given by

$$n(\epsilon) = \frac{g(\epsilon)}{e^{\beta\epsilon} - 1} = \frac{8\pi V}{h^3 c^3} \cdot \frac{\epsilon^2}{e^{\beta\epsilon} - 1}. \text{ (number of photons per unit energy range)}$$

The number of photons in the energy range $d\epsilon$ is therefore given by $n(\epsilon) \cdot d\epsilon$. The energy of the photons in the energy range $d\epsilon$ is therefore

$$[n(\epsilon) \cdot d\epsilon] \cdot \epsilon = \frac{8\pi V}{h^3 c^3} \cdot \frac{\epsilon^3}{e^{\beta\epsilon} - 1} \cdot d\epsilon.$$

Since for a photon with frequency ν, $\epsilon = h\nu$, the right-hand side of the above expression takes the form

$$[n(\epsilon) \cdot d\epsilon] \cdot \epsilon = \frac{8\pi\nu^2 V}{c^3} \cdot \frac{h\nu}{e^{\beta h\nu} - 1} \cdot d\nu.$$

Thus,

$$\boxed{\frac{[n(\epsilon) \cdot d\epsilon] \cdot \epsilon}{V \cdot d\nu} = \rho(\nu, T) = \frac{8\pi\nu^2}{c^3} \cdot \frac{h\nu}{\exp(\frac{h\nu}{k_B T}) - 1}, \text{ Planck's radiation law}}$$

where we have substituted $1/(k_B T)$ for β. We have thus derived Planck's radiation formula, where the meaning of $\rho(\nu, T)$ is clear: it is the energy density (which $[n(\epsilon) \cdot d\epsilon] \cdot \epsilon / V$ on the left-hand side signifies) of the electromagnetic radiation per unit frequency range (which $d\nu$ in the denominator on the left-hand side signifies) at frequency ν and at absolute temperature T. A note on the notion "per unit frequency range" is in order: If the left-hand side of Planck's radiation law is "per unit frequency range", then so must the right-hand side be. However, a frequency range ($d\nu$) does not appear on the right-hand side. This can often create confusion in the reader. This potential confusion can be remedied once it is recognized that $\rho(\nu, T)$ is an energy distribution function, and therefore, only the integral $\int_{\nu_1}^{\nu_2} \rho(\nu, T) d\nu$ carries significance. If we imagine volume V to be a uniform cylinder with unit cross sectional area and height c meters [the distance light travels in a vacuum in a second (unit time)], then $\int_{\nu_1}^{\nu_2} \rho(\nu, T) d\nu$ is the amount of radiant energy in the frequency interval $[\nu_1, \nu_2]$ crossing a unit area of space in unit time in a (vacuum) cavity whose walls are in thermal equilibrium with radiation at absolute temperature T.

We end with the following memorable quote: "The Planck function [radiation law] is worthy of respect, if not awe, in that it contains not one, not two, but *three* fundamental (or at least believed to be so) constants of nature: speed of light in a vacuum c, Planck's constant h, and Boltzmann's constant k_B.You can't get much more fundamental than that."[12]

[12] *Fundamentals of Atmospheric Radiation*, C. F. Bohren and E. E. Clothiaux, Wiley, 2006, p. 7.

A Approximating $\ln n!$ for Large n

For a non-negative integer n, $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$. Therefore,

$$\ln n! = \ln[n \times (n-1) \times (n-2) \times \cdots \times 1] = \ln n + \ln(n-1) + \ln(n-2) + \cdots + \ln 1 = \sum_{x=1}^{n} \ln x.$$

Consider now the continuous function $\ln x$ in the range $1 \leq x \leq n$. Let us a draw a rectangle of unit width (that is, $\Delta x = 1$) where its left vertical side has height $\ln x$. Then the sum of the areas of the rectangles in the range $1 \leq x \leq n$ will be equivalent to $\sum_1^n \ln x$. There is a gap between the top of a given rectangle and the envelope of the continuous curve $\ln x$, which diminishes as x increases. Thus, for large n, we can approximate $\sum_1^n \ln x$ (an area) by the integral $\int_1^n \ln x \; dx$ (also an area). Thus, for large n,

$$\ln n! = \sum_{1}^{n} \ln x \simeq \int_1^n \ln x \; dx, \quad \text{for large } n.$$

We can now utilize integration by parts to evaluate the right-hand side, where for functions u and v,

$$\int_a^b v \frac{du}{dx} \; dx = vu \Big|_a^b - \int_a^b u \frac{dv}{dx} \; dx.$$

Let us take $v = \ln x$, $u = x$, $a = 1$, and $b = n$. Then,

$$\int_1^n \ln x \frac{dx}{dx} \; dx = x \ln x \Big|_1^n - \int_1^n x \frac{d \ln x}{dx} \; dx = (n \ln n - 1 \ln 1) - \int_1^n x \frac{1}{x} \; dx = n \ln n - \int_1^n dx.$$

Thus,

$$\int_1^n \ln x \; dx = n \ln n - n + 1 \simeq n \ln n - n, \quad \text{for large } n.$$

Therefore, we obtain Stirling's approximation

$$\boxed{\ln n! \simeq n \ln n - n, \quad \text{for large } n.}$$

B Energy of the Quantum Harmonic Oscillator

Consider a harmonic oscillator with angular frequency ω. If the oscillator is in equilibrium with the electromagnetic field (photons), then, it can exchange energy with the field in discrete units of energy $\mathfrak{n}\hbar\omega$ where $\mathfrak{n} = 1, 2, \ldots$. This, essentially, is the Planck's hypothesis. The total energy of the harmonic oscillator is then given by $\mathfrak{n}\hbar\omega$ plus the energy of its ground-state. This ground-state energy, however, is not zero, since the Heisenberg's Uncertainty Principle prevents the oscillator from completely seizing motion.

We can hypothesize the ground-state of the oscillator as being in equilibrium, not with real photons but with virtual photons, which appear from and disappear into the void continuously. If we associate virtual photons with energy $\hbar\omega$ appearing $(+)$ from the vacuum and disappearing $(-)$ into the vacuum, then, at any instant, given that there is no restriction to the number of such photons, the total energy (E_{vac}) of the vacuum is

$$E_{\text{vac}} = \hbar\omega - \hbar\omega + \hbar\omega - \hbar\omega + \cdots = \hbar\omega(1 - 1 + 1 - 1 + \cdots).$$

If we consider pairwise sums $(1-1)+(1-1)+\cdots$, we may argue that the sum $(1-1+1-1+\cdots)$ amounts to 0. However, if we write $1-1+1-1+\cdots = 1-(1-1+1-1+\cdots) = 1-[(1-1)+(1-1)+\cdots] = 1-[0]$, then we may argue that the sum is 1. Rather than being paradoxical, the series sum $(1-1+1-1+\cdots)$ can be given a rigorous value, which is the average of 0 and 1. To see this, let the sum $1-1+1-1+\cdots = S$. Then,

$$S = 1-1+1-1+\cdots = 1-(1-1+1-1+\cdots) = 1-(S) \implies 2S = 1 \implies S = 1/2.$$

Substituting this result in the above expression for vacuum energy, we obtain

$$E_{\text{vac}} = \frac{1}{2}\hbar\omega.$$

Thus the virtual photons in equilibrium with the ground-state of the quantum harmonic oscillator has energy $\hbar\omega/2$, which in turn implies that the ground-state energy of the quantum harmonic oscillator itself is $\hbar\omega/2$. Therefore, the total energy of the quantum harmonic oscillator is given by

$$\boxed{E_{\mathfrak{n}} = \left(\mathfrak{n} + \frac{1}{2}\right)\hbar\omega, \quad \mathfrak{n} = 0, 1, 2, \ldots}$$

where now $\mathfrak{n}$ runs from 0 onwards. Note that this is a very heuristic argument; the reader can turn to a standard text on quantum mechanics for a rigorous derivation.